The Riemann-Roch Theorem

Göttingen Mathematical Institute

Paul Baum
Penn State

9 February, 2017

Five lectures:

1. Dirac operator✓
2. Atiyah-Singer revisited✓
3. What is K-homology?✓
4. The Riemann-Roch theorem
5. K-theory for group C^* algebras

THE RIEMANN-ROCH THEOREM

1. Classical Riemann-Roch✓
2. Hirzebruch-Riemann-Roch (HRR)
3. Grothendieck-Riemann-Roch (GRR)
4. RR for possibly singular varieties (Baum-Fulton-MacPherson)

REFERENCES

P. Baum, W. Fulton, and R. MacPherson *Riemann-Roch for singular varieties* Publ. Math. IHES 45: 101-167, 1975.

P. Baum, W. Fulton, and R. MacPherson *Riemann-Roch and topological K-theory for singular varieties* Acta Math. 143: 155-192, 1979.

P. Baum, W. Fulton, and G. Quart *Lefschetz-Riemann-Roch for singular varieties* Acta Math. 143: 193-211, 1979.

HIRZEBRUCH-RIEMANN-ROCH

M non-singular projective algebraic variety / $\mathbb{C}$

E an algebraic vector bundle on M

$\underline{E}$ = sheaf of germs of algebraic sections of E

$H^j(M, \underline{E}) :=$ j-th cohomology of M using $\underline{E}$, $j = 0, 1, 2, 3, \ldots$

LEMMA

For all $j = 0, 1, 2, \ldots \dim_{\mathbb{C}} H^j(M, \underline{E}) < \infty$.

For all $j > \dim_{\mathbb{C}}(M), \quad H^j(M, \underline{E}) = 0$.

$$\chi(M, E) := \sum_{j=0}^{n} (-1)^j \dim_{\mathbb{C}} H^j(M, \underline{E})$$

$n = \dim_{\mathbb{C}}(M)$

THEOREM[HRR] Let M be a non-singular projective algebraic variety / $\mathbb{C}$ and let E be an algebraic vector bundle on M. Then

$$\chi(M, E) = (ch(E) \cup Td(M))[M]$$

Hirzebruch-Riemann-Roch

Theorem (HRR)

Let M be a non-singular projective algebraic variety $/\,\mathbb{C}$ and let E be an algebraic vector bundle on M. Then

$$\chi(M,E) = (ch(E) \cup Td(M))[M]$$

EXAMPLE. Let M be a compact complex-analytic manifold.
Set $\Omega^{p,q} = C^\infty(M, \Lambda^{p,q}T^*M)$
$\Omega^{p,q}$ is the $\mathbb{C}$ vector space of all C^∞ differential forms of type (p,q)
Dolbeault complex

$$0 \longrightarrow \Omega^{0,0} \longrightarrow \Omega^{0,1} \longrightarrow \Omega^{0,2} \longrightarrow \cdots \longrightarrow \Omega^{0,n} \longrightarrow 0$$

The Dirac operator (of the underlying Spin^c manifold) is the assembled Dolbeault complex

$$\bar\partial + \bar\partial^* \colon \bigoplus_j \Omega^{0,\,2j} \longrightarrow \bigoplus_j \Omega^{0,\,2j+1}$$

The index of this operator is the arithmetic genus of M — i.e. is the Euler number of the Dolbeault complex.

Let X be a finite CW complex.
The three versions of K-homology are isomorphic.

$$K_j^{homotopy}(X) \underset{\longleftarrow}{\overset{\longrightarrow}{}} K_j(X) \longrightarrow KK^j(C(X), \mathbb{C})$$

homotopy theory K-cycles Atiyah-BDF-Kasparov

$$j = 0, 1$$

Let X be a finite CW complex.
The three versions of K-homology are isomorphic.

$$K_j^{homotopy}(X) \underset{\longleftarrow}{\overset{\longrightarrow}{}} K_j(X) \longrightarrow KK^j(C(X), \mathbb{C})$$

homotopy theory K-cycles   Atiyah-BDF-Kasparov

$$j = 0, 1$$

X is a finite CW complex.

CHERN CHARACTER

The Chern character is often viewed as a functorial map of contravariant functors :

$$ch\colon K^j(X) \longrightarrow \bigoplus_l H^{j+2l}(X;\mathbb{Q})$$

$$j = 0, 1$$

Note that this is a map of rings.

X is a finite CW complex.

A more inclusive (and more accurate) view of the Chern character is that it is a pair of functorial maps :

$$ch\colon K^j(X) \longrightarrow \bigoplus_l H^{j+2l}(X;\mathbb{Q}) \qquad \textcolor{red}{contravariant}$$

$$ch_{\#}\colon K_j(X) \longrightarrow \bigoplus_l H_{j+2l}(X;\mathbb{Q}) \qquad \textcolor{red}{covariant}$$

$K_*(X)$ is a module over $K^*(X)$.

$H_*(X;\mathbb{Q})$ is a module over $H^*(X;\mathbb{Q})$. cap product

The Chern character respects these module structures.

Definition of the Chern character in homology $\quad j = 0, 1$

$$ch_{\#} \colon K_j(X) \longrightarrow \bigoplus_l H_{j+2l}(X; \mathbb{Q}) \ \ \textit{covariant}$$

$$ch_{\#}(M, E, \varphi) := \varphi_*(ch(E) \cup Td(M) \cap [M])$$

$$\varphi_* \colon H_*(M; \mathbb{Q}) \longrightarrow H_*(X; \mathbb{Q})$$

$ch(E) \cup Td(M) \cap [M] :=$ Poincare dual of $ch(E) \cup Td(M)$

$K_*(X)$ is a module over $K^*(X)$.

Let (M, E, φ) be a K-cycle on X.
Let F be a $\mathbb{C}$ vector bundle on X.

Then:

$$F \cdot (M, E, \varphi) := (M, E \otimes \varphi^*(F), \varphi)$$

and the module structure is respected :

$$ch_{\#}(F \cdot (M, E, \varphi)) = ch(F) \cap ch_{\#}(M, E, \varphi)$$

K-theory and K-homology in algebraic geometry

Let X be a (possibly singular) projective algebraic variety $/\mathbb{C}$.

Grothendieck defined two abelian groups:

$K^0_{alg}(X) =$ Grothendieck group of algebraic vector bundles on X.

$K_0^{alg}(X) =$ Grothendieck group of coherent algebraic sheaves on X.

$K^0_{alg}(X) =$ the algebraic geometry K-theory of X contravariant.

$K_0^{alg}(X) =$ the algebraic geometry K-homology of X covariant.

K-theory in algebraic geometry

$\mathsf{Vect}_{alg}X =$
set of isomorphism classes of algebraic vector bundles on X.

$\mathsf{A}(\mathsf{Vect}_{alg}X) =$ free abelian group
with one generator for each element $[E] \in \mathrm{Vect}_{alg}X$.

For each short exact sequence ξ

$$0 \to E' \to E \to E'' \to 0$$

of algebraic vector bundles on X, let $r(\xi) \in \mathrm{A}(\mathrm{Vect}_{alg}X)$ be

$$r(\xi) := [E'] + [E''] - [E]$$

K-theory in algebraic geometry

$\mathcal{R} \subset \mathrm{A}(\mathrm{Vect}_{alg}(X))$ is the subgroup of $\mathrm{A}(\mathrm{Vect}_{alg}X)$ generated by all $r(\xi) \in \mathrm{A}(\mathrm{Vect}_{alg}X)$.

DEFINITION. $K^0_{alg}(X) := \mathrm{A}(\mathrm{Vect}_{alg}X)/\mathcal{R}$

Let X, Y be (possibly singular) projective algebraic varieties $/\mathbb{C}$.
Let

$$f\colon X \longrightarrow Y$$

be a morphism of algebraic varieties.
Then have the map of abelian groups

$$f^*\colon K^0_{alg}(X) \longleftarrow K^0_{alg}(Y)$$

$$[f^*E] \leftarrow [E]$$

Vector bundles pull back. f^*E is the pull-back via f of E.

K-homology in algebraic geometry

$\mathcal{S}_{alg}X =$
set of isomorphism classes of coherent algebraic sheaves on X.

$\mathsf{A}(\mathcal{S}_{alg}X) =$ free abelian group
with one generator for each element $[\mathcal{E}] \in \mathcal{S}_{alg}X$.

For each short exact sequence ξ

$$0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$$

of coherent algebraic sheaves on X, let $r(\xi) \in \mathrm{A}(\mathcal{S}_{alg}X)$ be

$$r(\xi) := [\mathcal{E}'] + [\mathcal{E}''] - [\mathcal{E}]$$

K-homology in algebraic geometry

$\mathfrak{R} \subset \mathrm{A}(\mathcal{S}_{alg}(X))$ is the subgroup of $\mathrm{A}(\mathcal{S}_{alg}X)$ generated by all $r(\xi) \in \mathrm{A}(\mathcal{S}_{alg}X)$.

DEFINITION. $K_0^{alg}(X) := \mathrm{A}(\mathcal{S}_{alg}X)/\mathfrak{R}$

Let X, Y be (possibly singular) projective algebraic varieties $/\mathbb{C}$. Let

$$f\colon X \longrightarrow Y$$

be a morphism of algebraic varieties.
Then have the map of abelian groups

$$f_*\colon K_0^{alg}(X) \longrightarrow K_0^{alg}(Y)$$

$$[\mathcal{E}] \mapsto \Sigma_j (-1)^j [(R^j f)\mathcal{E}]$$

$f\colon X \to Y$ morphism of algebraic varieties
$\mathcal{E}$ coherent algebraic sheaf on X
For $j \geq 0$, define a presheaf $(W^j f)\mathcal{E}$ on Y by

$$U \mapsto H^j(f^{-1}U;\, \mathcal{E}|f^{-1}U) \qquad\qquad U \text{ an open subset of } Y$$

Then

$$(R^j f)\mathcal{E} := \text{the sheafification of } (W^j f)\mathcal{E}$$

$f\colon X \to Y \quad$ morphism of algebraic varieties

$$f_*\colon K_0^{alg}(X) \longrightarrow K_0^{alg}(Y)$$

$$[\mathcal{E}] \mapsto \Sigma_j (-1)^j [(R^j f)\mathcal{E}]$$

SPECIAL CASE of $f_*\colon K_0^{alg}(X) \longrightarrow K_0^{alg}(Y)$

Y is a point. $Y = \cdot$
$\epsilon\colon X \to \cdot$ is the map of X to a point.

$K^0_{alg}(\cdot) = K_0^{alg}(\cdot) = \mathbb{Z}$

$\epsilon_*\colon K_0^{alg}(X) \to \ K_0^{alg}(\cdot) = \mathbb{Z}$

$\epsilon_*(\mathcal{E}) = \chi(X;\mathcal{E}) = \Sigma_j(-1)^j \mathrm{dim}_{\mathbb{C}} H^j(X;\mathcal{E})$

X non-singular $\Longrightarrow K^0_{alg}(X) \cong K^{alg}_0(X)$

Let X be non-singular.
Let E be an algebraic vector bundle on X.
$\underline{E}$ denotes the sheaf of germs of algebraic sections of E.
Then $E \mapsto \underline{E}$ is an isomorphism of abelian groups

$$K^0_{alg}(X) \longrightarrow K^{alg}_0(X)$$

This is Poincaré duality within the context of algebraic geometry K-theory&K-homology.

X non-singular $\Longrightarrow K^0_{alg}(X) \cong K_0^{alg}(X)$

Let X be non-singular.

The inverse map

$$K_0^{alg}(X) \to K^0_{alg}(X)$$

is defined as follows.

Let $\mathcal{F}$ be a coherent algebraic sheaf on X.
Since X is non-singular,
$\mathcal{F}$ has a finite resolution by algebraic vector bundles.

X non-singular $\Longrightarrow K^0_{alg}(X) \cong K_0^{alg}(X)$

$\mathcal{F}$ has a finite resolution by algebraic vector bundles.
i.e. $\exists$ algebraic vector bundles on X $E_r, E_{r-1}, \ldots, E_0$ and an exact sequence of coherent algebraic sheaves

$$0 \to \underline{E_r} \to \underline{E_{r-1}} \to \ldots \to \underline{E_0} \to \mathcal{F} \to 0$$

Then $K_0^{alg}(X) \to K^0_{alg}(X)$ is

$$\mathcal{F} \mapsto \Sigma_j (-1)^j \, E_j$$

Grothendieck-Riemann-Roch

Theorem (GRR)

Let X, Y *be non-singular projective algebraic varieties* $/\mathbb{C}$ *, and let* $f : X \longrightarrow Y$ *be a morphism of algebraic varieties. Then there is commutativity in the diagram :*

$$\begin{array}{ccc} & K^0_{alg}(X) \longrightarrow K^0_{alg}(Y) & \\ ch(\ ) \cup Td(X) \quad \downarrow & & \downarrow \quad ch(\ ) \cup Td(Y) \\ & H^*(X;\mathbb{Q}) \longrightarrow H^*(Y;\mathbb{Q}) & \end{array}$$

WARNING!!!

The horizontal arrows in the GRR commutative diagram

$$\begin{array}{ccc} & K^0_{alg}(X) \longrightarrow K^0_{alg}(Y) & \\ ch(\,) \cup Td(X) & \downarrow \qquad\qquad \downarrow & ch(\,) \cup Td(Y) \\ & H^*(X;\mathbb{Q}) \longrightarrow H^*(Y;\mathbb{Q}) & \end{array}$$

are wrong-way (i.e. Gysin) maps.

$$K^0_{alg}(X) \cong K_0^{alg}(X) \xrightarrow{f_*} K_0^{alg}(Y) \cong K^0_{alg}(Y)$$

$$H^*(X;\mathbb{Q}) \cong H_*(X;\mathbb{Q}) \xrightarrow{f_*} H_*(Y;\mathbb{Q}) \cong H^*(Y;\mathbb{Q})$$

Poincaré duality Poincaré duality

Riemann-Roch for possibly singular complex projective algebraic varieties

Let X be a (possibly singular) projective algebraic variety / $\mathbb{C}$

Then (Baum-Fulton-MacPherson) there are functorial maps

$\alpha_X \colon K^0_{alg}(X) \longrightarrow K^0_{top}(X)$ K-theory contravariant
natural transformation of contravariant functors

$\beta_X \colon K_0^{alg}(X) \longrightarrow K_0^{top}(X)$ K-homology covariant
natural transformation of covariant functors

Everything is natural. No wrong-way (i.e. Gysin) maps are used.

$\alpha_X \colon K^0_{alg}(X) \longrightarrow K^0_{top}(X)$
is the forgetful map which sends an algebraic vector bundle E to the underlying topological vector bundle of E.

$$\alpha_X(E) := E_{\text{topological}}$$

Let X, Y be projective algebraic varieties $/\mathbb{C}$, and let $f : X \longrightarrow Y$ be a morphism of algebraic varieties. Then there is commutativity in the diagram :

$$\begin{array}{ccc} K^0_{alg}(X) & \longleftarrow & K^0_{alg}(Y) \\ \alpha_X \downarrow & & \downarrow \alpha_Y \\ K^0_{top}(X) & \longleftarrow & K^0_{top}(Y) \end{array}$$

i.e. natural transformation of contravariant functors

Let X, Y be projective algebraic varieties $/\mathbb{C}$, and let $f : X \longrightarrow Y$ be a morphism of algebraic varieties. Then there is commutativity in the diagram :

$$\begin{array}{ccc} K^0_{alg}(X) & \longleftarrow & K^0_{alg}(Y) \\ \alpha_X \downarrow & & \downarrow \alpha_Y \\ K^0_{top}(X) & \longleftarrow & K^0_{top}(Y) \\ ch \downarrow & & \downarrow ch \\ H^*(X;\mathbb{Q}) & \longleftarrow & H^*(Y;\mathbb{Q}) \end{array}$$

Let X, Y be projective algebraic varieties $/\mathbb{C}$, and let $f : X \longrightarrow Y$ be a morphism of algebraic varieties. Then there is commutativity in the diagram :

$$\begin{array}{ccc} K_0^{alg}(X) & \longrightarrow & K_0^{alg}(Y) \\ \beta_X \downarrow & & \downarrow \beta_Y \\ K_0^{top}(X) & \longrightarrow & K_0^{top}(Y) \end{array}$$

i.e. natural transformation of covariant functors

<u>Notation.</u> K_*^{top} is K-cycle K-homology.

Let X, Y be projective algebraic varieties $/\mathbb{C}$, and let $f : X \longrightarrow Y$ be a morphism of algebraic varieties. Then there is commutativity in the diagram :

$$\begin{array}{ccc} K^0_{alg}(X) & \longleftarrow & K^0_{alg}(Y) \\ \alpha_X \downarrow & & \downarrow \alpha_Y \\ K^0_{top}(X) & \longleftarrow & K^0_{top}(Y) \\ ch \downarrow & & \downarrow ch \\ H^*(X;\mathbb{Q}) & \longleftarrow & H^*(Y;\mathbb{Q}) \end{array}$$

Let X, Y be projective algebraic varieties $/\mathbb{C}$, and let $f : X \longrightarrow Y$ be a morphism of algebraic varieties. Then there is commutativity in the diagram :

$$\begin{array}{ccc} K_0^{alg}(X) & \longrightarrow & K_0^{alg}(Y) \\ \beta_X \downarrow & & \downarrow \beta_Y \\ K_0^{top}(X) & \longrightarrow & K_0^{top}(Y) \\ ch_{\#} \downarrow & & \downarrow ch_{\#} \\ H_*(X;\mathbb{Q}) & \longrightarrow & H_*(Y;\mathbb{Q}) \end{array}$$

Definition of $\beta_X \colon K_0^{alg}(X) \to K_0^{top}(X)$

Let $\mathcal{F}$ be a coherent algebraic sheaf on X.

Choose an embedding of projective algebraic varieties

$$\iota \colon X \hookrightarrow W$$

where W is non-singular.

$\iota_*\mathcal{F}$ is the push forward (i.e. extend by zero) of $\mathcal{F}$.
$\iota_*\mathcal{F}$ is a coherent algebraic sheaf on W.

$\iota_* \mathcal{F}$ is a coherent algebraic sheaf on W.
Since W is non-singular, $\iota_* \mathcal{F}$ has a finite resolution by algebraic vector bundles.

$$0 \to \underline{E_r} \to \underline{E_{r-1}} \to \ldots \to \underline{E_0} \to \iota_* \mathcal{F} \to 0$$

Consider

$$0 \to E_r \to E_{r-1} \to \ldots \to E_0 \to 0$$

These are algebraic vector bundles on W and maps of algebraic vector bundles such that for each $p \in W - \iota(X)$ the sequence of finite dimensional $\mathbb{C}$ vector spaces

$$0 \to (E_r)_p \to (E_{r-1})_p \to \ldots \to (E_0)_p \to 0$$

is exact.

Choose Hermitian structures for $E_r, E_{r-1}, \ldots, E_0$
Then for each vector bundle map

$$\sigma \colon E_j \to E_{j-1}$$

there is the adjoint map

$$\sigma^* \colon E_j \leftarrow E_{j-1}$$

$$\sigma \oplus \sigma^* \colon \bigoplus_j E_{2j} \longrightarrow \bigoplus_j E_{2j+1}$$

is a map of topological vector bundles which is an isomorphism on $W - \iota(X)$.

Let Ω be an open set in W with smooth boundary $\partial\Omega$ such that $\overline{\Omega} = \Omega \cup \partial\Omega$ is a compact manifold with boundary which retracts onto $\iota(X)$. $\overline{\Omega} \to \iota(X)$.
Set

$$M = \overline{\Omega} \cup_{\partial\Omega} \overline{\Omega}$$

M is a closed Spin^c manifold which maps to X by:

$$\varphi\colon M = \overline{\Omega} \cup_{\partial\Omega} \overline{\Omega} \to \overline{\Omega} \to \iota(X) = X$$

On $M = \overline{\Omega} \cup_{\partial\Omega} \overline{\Omega}$ let E be the topological vector bundle

$$E = \bigoplus_j E_{2j} \cup_{(\sigma \oplus \sigma^*)} \bigoplus_j E_{2j+1}$$

Then $\beta_X \colon K_0^{alg}(X) \to K_0^{top}(X)$ is :

$$\mathcal{F} \mapsto (M, E, \varphi)$$

$$M = \overline{\Omega} \cup_{\partial\Omega} \overline{\Omega}$$

Equivalent definition of $\beta_X\colon K_0^{alg}(X) \to K_0^{top}(X)$

Let (M, E, φ) be an algebraic K-cycle on X, i.e.

- M is a non-singular complex projective algebraic variety.
- E is an algebraic vector bundle on M.
- $\varphi\colon M \to X$ is a morphism of projective algebraic varieties.

Then:

$$\beta_X(\varphi_*(\underline{E})) = (M, E, \varphi)_{\text{topological}}$$

Module structure

$K^0_{alg}(X)$ is a ring and $K_0^{alg}(X)$ is a module over this ring.

$\alpha_X\colon K^0_{alg}(X) \to K^0_{top}(X)$ is a homomorphism of rings.

$\beta_X\colon K_0^{alg}(X) \to K_0^{top}(X)$ respects the module structures.

Todd class

Set

$$\mathrm{td}(X) = \mathrm{ch}_{\#}\left(\beta_X(\mathcal{O}_X)\right) \qquad \mathrm{td}(X) \in H_*(X;\mathbb{Q})$$

If X is non-singular, then $\mathrm{td}(X) = \mathrm{Todd}(X) \cap [X]$.

With X possibly singular and E an algebraic vector bundle on X

$$\chi(X, \underline{E}) = \epsilon_*(\mathrm{ch}(E) \cap \mathrm{td}(X))$$

$\epsilon \colon X \to \cdot$ is the map of X to a point.

$\epsilon_* \colon H_*(X;\mathbb{Q}) \to \ H_*(\cdot\ ;\mathbb{Q}) = \mathbb{Q}$

Let

$$\begin{array}{c} \widetilde{X} \\ \Big\downarrow \pi \\ X \end{array}$$

be resolution of singularities in the sense of Hironaka.

$\pi_*\colon H_*(\widetilde{X};\mathbb{Q}) \to H_*(X;\mathbb{Q})$

<u>Lemma.</u> $\pi_*(Td(\widetilde{X}) \cap [\widetilde{X}])$ is intrinsic to X i.e. does not depend on the choice of the resolution of singularities.

$td(X) \in H_*(X : \mathbb{Q})$ is also intrinsic to X.

$td(X) - \pi_*(Td(\widetilde{X}) \cap [\widetilde{X}])$ is given by a homology class on X which (in a canonical way) is supported on the singular locus of X.

Problem. In examples calculate $td(X) \in H_*(X; \mathbb{Q})$.

For toric varieties see papers of J. Shaneson and S. Cappell.