Random Schrödinger Operators

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Motivation

Goal : to study the electronic transport in disordered materials and identify if a material is **a conductor or an insulator**

Quantum mechanics setting :

physical state	a vector ψ in a Hilbert space $\mathcal{H}$, with $\|\psi\| = 1$
physical observables	self-adjoint operator H
possible outcomes	$\sigma(H)$ spectrum of the operator H

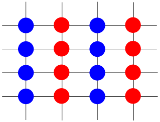
Dynamics of a particle moving in a material : $\psi \in \mathcal{H} = L^2(\mathbb{R}^d)$ or $\ell^2(\mathbb{Z}^d)$, $\|\psi\| = 1$,

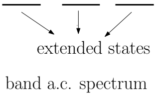
$$\partial_t \psi(t,x) = -iH\psi(t,x),$$

$$\psi(t,x) = e^{-itH}\psi(0,x),$$

where $H = H_0 + V$ is a self-adjoint Schrödinger operator on $\mathcal{H}$.

Example : electrons in a crystal, $H = -\Delta + V$ acting on $\ell^2(\mathbb{Z}^d)$, the potential $V\psi(x) = q(x)\psi(x)$, where q is a periodic function.





extended states $\sim \psi(t,x)$ propagate in space as t grows $\sim$ transport

Dynamics of a particle moving in a material : $\psi \in \mathcal{H} = L^2(\mathbb{R}^d)$ or $\ell^2(\mathbb{Z}^d)$, $\|\psi\| = 1$,

$$\partial_t \psi(t,x) = -iH\psi(t,x),$$

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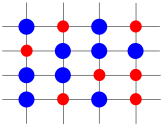
where $H = H_0 + V$ is a self-adjoint Schrödinger operator on $\mathcal{H}$.

Example : electrons in a disordered crystal



$\psi(t,x)$ do not propagate in space as t grows $\sim$ absence of transport

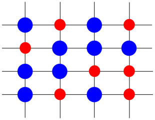
Disordered media

P. W. Anderson 1958 :
if the medium has impurities, there is *no wave propagation*.
"Absence of diffusion in certain random lattices", Phys. Rev. (Nobel 1977)

Anderson model : $H_\omega = -\Delta + V_\omega$ on $\ell^2(\mathbb{Z}^d)$, with

$$V_\omega(x) = \sum_{j \in \mathbb{Z}^d} \omega_j \delta_j(x),$$

where $\omega = (\omega_j)_{j \in \mathbb{Z}^d}$ is a random variable in a probability space $(\Omega, \mathbb{P})$.



Localization : first rigorous mathematical results in the late 70s, early 80s.

Recall from spectral theory

For a self-adjoint operator H and a vector $\varphi \in \mathcal{H}$, there exists a spectral measure $\mu_{H,\varphi}$ such that

$$\langle \varphi, H\varphi \rangle = \int_{\mathbb{R}} \lambda d\mu_{H,\varphi}(\lambda)$$

or, formally

$$H = \int_{\mathbb{R}} \lambda d\mu_{H,\varphi}(\lambda).$$

For this spectral measure $\mu = \mu_{H,\varphi}$ one has the usual Lebesgue decomposition into three mutually singular parts

$$\mu = \mu^{pp} + \mu^{sc} + \mu^{ac}$$

which induces a decomposition of the Hilbert space $\mathcal{H} = \mathcal{H}_{pp} \oplus \mathcal{H}_{sc} \oplus \mathcal{H}_{ac}$, such that

$$H_{\mathcal{H}_*} = \int_{\mathbb{R}} \lambda d\mu^*_{H,\varphi}(\lambda), \quad * \in pp, sc, ac$$

Then, writing

$$\sigma_*(H) = \sigma(H_{\mathcal{H}_*}), \quad * \in pp, sc, ac$$

we have the following decomposition for the spectrum

$$\sigma(H) = \sigma_{pp}(H) \cup \sigma_{sc}(H) \cup \sigma_{ac}(H)$$

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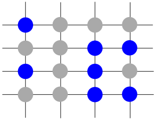
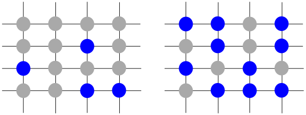
The Anderson Model : on each point of the lattice we place a potential, which can be • or •.



We consider many possible configurations. Every configuration of the potential is a vector ω in a probability space $(\Omega, \mathbb{P})$.

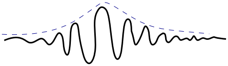


We get a random operator $\omega \mapsto H_\omega = -\Delta + V_\omega$, where

$$V_\omega(x) = \sum_{j \in \mathbb{Z}^d} \omega_j \delta_j(x),$$

with $\omega_j \in \{\bullet, \bullet\}$ bounded, independent, identically distributed random variables.

For typical ω, $\psi_\omega(t, x)$ does not propagate in space as t grows $\sim$ absence of transport



Anderson localization (disambiguation)

Consider the Anderson model $H_\omega = -\Delta + V_\omega$ acting on a Hilbert space $\mathcal{H}$.
We say it exhibits :

- *spectral localization* in an interval I if $\sigma(H) \cap I = \sigma_{pp}(H) \cap I$, almost surely.
- *Anderson localization* (AL) in I if $\sigma(H) \cap I = \sigma_{pp}(H) \cap I$ with exponentially decaying eigenfunctions, almost surely.
- *dynamical localization* (DL) in I if there exist constants $C < \infty$ and $c > 0$ such that for all $x, y \in \mathbb{Z}^d$,

$$(DL) \qquad \mathbb{E}\left(\sup_{t\in\mathbb{R}} |\langle \delta_y, e^{-itH_\omega} \chi_I(H_\omega)\delta_x\rangle|\right) \leq Ce^{-c|x-y|}$$

- delocalization in I when (DL) does not hold.

dyn. loc $\Rightarrow$ Anderson loc. $\Rightarrow$ spectral loc.
dyn. loc $\not\Leftarrow$ Anderson loc. $\not\Leftarrow$ spectral loc.

Consequences of dynamical localization in an interval I

- Absence of transport If (DL) holds in $I \subset \mathbb{R}$, then for $\varphi \in \ell^2(\mathbb{Z}^d)$ with compact support we have

weighted space

time evolution

$$\sup_t \| |X|^{p/2} e^{-itH_\omega} \chi_J(H_\omega) \varphi \| < \infty,$$

restriction in energy

for every $p \geq 0$, with probability one.

- In particular, $\sup_t \langle X^2 \rangle_I(t) < \infty$ almost surely.
- Decay of Fermi projector kernel : if $E \in I$, there exist constants $C_1 < \infty$ and $C_2 > 0$ such that

$$\mathbb{E}\left(|\langle \delta_y, \chi_{(-\infty,E)}(H_\omega)\delta_x \rangle| \right) \leq C_1 e^{-C_2|x-y|}$$

What is known

Consider the operator $H_\omega = -\Delta + \lambda V_\omega$, with $\lambda > 0$ acting on $\ell^2(\mathbb{Z}^d)$.

Theorem (Localization in $d = 1$)

For any $\lambda > 0$ H_ω exhibits localization throughout its spectrum a.s.

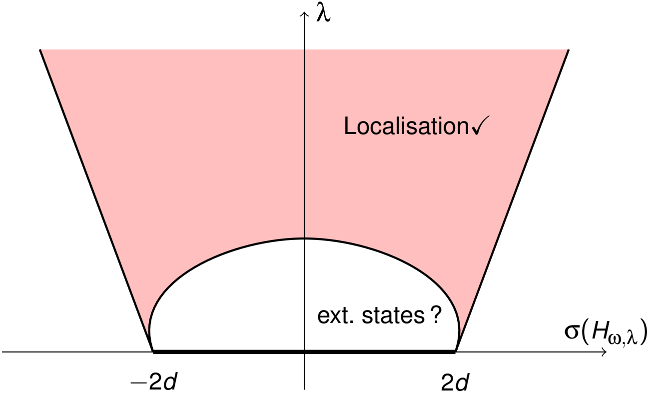
Theorem (Localization $d > 1$)

i. *for $\lambda > 0$ large enough, H_ω exhibits localization throughout its spectrum a.s.*

ii. *for fixed λ, H_ω exhibits localization in intervals I at spectral edges a.s.*

Phase diagram for $H_{\omega,\lambda}$ on $\ell^2(\mathbb{Z}^d)$, with $d > 1$

Transport (Anderson) transition : passage from *localized* to *extended states*.



Remark : delocalization is an open problem !

We have now two tasks :

Determine the spectrum

Prove localization

Methods to prove localization in arbitrary dimension combine functional analysis and probability tools to show *the decay of resolvents*,

- Multiscale Analysis (Fröhlich-Spencer'83).
- Fractional Moment Method (Aizenman-Molchanov'93).

The Anderson model

Ergodic properties and spectrum

Some definitions from probability

- We consider a probability space $(\Omega, \mathcal{B}, \mathbb{P})$, where $\mathcal{B}$ is a σ-algebra and $\mathbb{P}$ is a probability measure on $(\Omega, \mathcal{B})$.
- Given a probability space $(\Omega, \mathcal{B}, \mathbb{P})$, a random variable is a measurable function $X : \Omega \to \mathbb{R}$.
- The probability distribution of X is the measure μ defined by

$$\mu(A) = \mathbb{P}(\{\omega \in \Omega; X(\omega) \in A\}).$$

- The support of the measure μ is given by

$$\mathrm{supp}\,\mu := \{x \in \mathbb{R}; \mu([x-\varepsilon, x+\varepsilon]) > 0, \forall \varepsilon > 0\}.$$

- If for any $A \in \mathcal{B}$, $\mathbb{P}(Y(\omega) \in A) = \mathbb{P}(X(\omega) \in A) = \mu(A)$, we say X and Y are *identically distributed*.
- A collection of random variables $\{X_i\}_{i \in \mathbb{Z}^d}$ is called a *stochastic process*.

- A collection of random variables $\{X_n\}$ is called *independent* if, for any finite subset $\{n_1, \ldots n_k\} \subset \mathbb{Z}^d$ and abritrary Borel sets $A_1, \ldots, A_k \subset \mathbb{R}$,

$$\mathbb{P}(X_{n_1}(\omega) \in A_1, \ldots, X_{n_k}(\omega) \in A_k) = \prod_{j=1}^{k} \mathbb{P}(X_{n_j}(\omega) \in A_j).$$

- If the collection of random variables $\{X_n\}$ is independent and identically distributed (i.i.d.), we have

$$\mathbb{P}(X_1(\omega) \in A, \ldots, X_k(\omega) \in A) = \prod_{j=1}^{k} \mu(A).$$

- We will often consider $(\Omega, \mathcal{B}, \mathbb{P}) = \left(\mathbb{R}^{\mathbb{Z}^d}, \mathcal{B}_{\mathbb{R}}, \bigotimes_{n \in \mathbb{Z}^d} \mu\right)$, where $\mathbb{R}^{\mathbb{Z}^d} := \bigotimes_{j \in \mathbb{Z}^d} \mathbb{R}$ and write $\omega := (\omega_n)_{n \in \mathbb{Z}^d}$ instead of $\{X_n(\omega)\}_{n \in \mathbb{Z}^d}$.

The Anderson model

$$H_\omega = -\Delta + \sum_{j \in \mathbb{Z}^d} \omega_j P_{\delta_j} \quad \text{on } \ell^2(\mathbb{Z}^d),$$

where $P_{\delta_j} = \langle \delta_j, \cdot \rangle \delta_j$.

- $-\Delta$ is the discrete Laplacian

$$-\Delta \varphi(n) = -\sum_{m \sim n} (\varphi(m) - \varphi(n)),$$

- ω_j are i.i.d. random variables, with probability distribution μ with compact support $\mathbb{A}$.
- $\Omega := \mathbb{A}^{\mathbb{Z}^d} \ni \omega := (\omega_j)$. The probability space is the product space $(\Omega, \mathcal{B}, \mathbb{P})$ with the product σ-algebra of Borel sets $\mathcal{B}$ and the product probability measure

$$\mathbb{P} = \bigotimes_{j \in \mathbb{Z}^d} \mu.$$

Analogously, we can define the Anderson model on $\ell^2(\Gamma)$, for Γ a countable set. For ex., on a tree with branching number K, called the Bethe lattice $\mathbb{B}$.

The Anderson model

$$H_\omega = -\Delta + \underbrace{\sum_{j\in\mathbb{Z}^d} \omega_j P_{\delta_j}}_{V_\omega} \quad \text{on } \ell^2(\mathbb{Z}^d),$$

where $P_{\delta_j} = \langle \delta_j, \cdot \rangle \delta_j$. This operator acts in the following way

$$\begin{aligned}(H_\omega \varphi)(n) &= -\Delta\varphi + V_\omega(n)\varphi(n) \\ &= -\Delta\varphi + \omega_n \varphi(n).\end{aligned}$$

Since $\operatorname{supp}\mu$ is compact, the potential V_ω is bounded. Moreover, V_ω is self-adjoint on $\ell^2(\mathbb{Z}^d)$.

Since $-\Delta$ and V_ω are self-adjoint, the operator $H_\omega = -\Delta + V_\omega$ is self-adjoint in $\ell^2(\mathbb{Z}^d)$.

Definition

The map $\Omega \ni \omega \mapsto H_\omega \in \mathcal{B}(\mathcal{H})$ is measurable if for any $\varphi, \psi \in \mathcal{H}$, the map $\Omega \ni \omega \mapsto \langle \varphi, H_\omega \psi \rangle \in \mathbb{C}$ is measurable.

If the operator is unbounded, we check measurability of $f(H_\omega)$, for any bounded function $f : \mathbb{R} \to \mathbb{C}$.

- The Anderson model $\omega \mapsto H_\omega$ on $\ell^2(\mathbb{Z}^d)$ is measurable.

Note that H_ω represents the *family* of operators $(H_\omega)_{\omega \in \Omega}$.

Definition

H_ω is called *ergodic* if there exists an ergodic group of transformations $(\tau_\gamma)_{\gamma\in\Gamma}$ acting on Ω associated to a family of unitary operators $(U_\gamma)_{\gamma\in\Gamma}$ on $\mathcal{H}$ s.t.

$$H_{\tau_\gamma(\omega)} = U_\gamma H_\omega U_\gamma^* \quad \text{for all } \gamma \in \Gamma.$$

• The Anderson model H_ω on $\ell^2(\mathbb{Z}^d)$ is ergodic with respect to $\mathbb{Z}^d$. That is, with respect to the translations $\tau_\gamma(\omega) = (\omega_{n+\gamma})_{n\in\mathbb{Z}^d}$ and $U_\gamma\varphi(n) = \varphi(n-\gamma)$ with $\gamma \in \mathbb{Z}^d$.

Ex. The Anderson model H_ω on $\ell^2(\mathbb{B})$, where $\mathbb{B}$ is the Bethe lattice under some constraints, is ergodic w.r.t. a certain family of transformations in $\mathbb{B}$ (see Acosta-Klein'92).

• The Anderson model H_ω on $\ell^2(\mathbb{Z}^d)$ is ergodic with respect to $\mathbb{Z}^d$.
Indeed, recall the family $\{\tau_\gamma\}_{\gamma\in\mathbb{Z}^d}$ of translations on Ω given by

$$\tau_\gamma(\omega) = (\omega_{n-\gamma})_{n\in\mathbb{Z}^d},$$

and the family of unitary operators U_γ acting on $\ell^2(\mathbb{Z}^d)$ defined by

$$U_\gamma\varphi(n) = \varphi(n-\gamma), \quad \gamma\in\mathbb{Z}^d.$$

Note that U_γ^* is given by $U_\gamma^*\varphi(n) = \varphi(n+\gamma) = U_{-\gamma}$. Then

$$\begin{aligned}
U_\gamma H_\omega U_{-\gamma}\varphi(n) &= U_\gamma(-\Delta)U_{-\gamma}\varphi(n) + U_\gamma\left(V_\omega U_{-\gamma}\right)\varphi(n)\\
&= -\Delta\varphi(n) + \left(V_\omega U_{-\gamma}\varphi\right)(n-\gamma)\\
&= -\Delta\varphi(n) + V_\omega(n-\gamma)\left(U_{-\gamma}\varphi\right)(n-\gamma)\\
&= -\Delta\varphi(n) + V_\omega(n-\gamma)\varphi(n).
\end{aligned}$$

Recall that V_ω acts in the following way : $V_\omega\varphi(n) = \omega_n\varphi(n)$, for all $n\in\mathbb{Z}^d$.
Therefore $V_\omega(n-\gamma)\varphi(n) = \omega_{n-\gamma}\varphi(n) = V_{\tau_\gamma(\omega)}\varphi(n)$, and so

$$U_\gamma H_\omega U_{-\gamma}\varphi = H_{\tau_\gamma(\omega)}.$$

□

Almost sure spectrum

Theorem (Pastur'80, Kunz-Souillard'80, Kirsch-Martinelli '82)

If H_ω is an ergodic operator, there exist closed sets Σ, $\Sigma_{pp}, \Sigma_{ac}, \Sigma_{sc} \subset \mathbb{R}$ such that for $\mathbb{P}$-a.e. $\omega \in \Omega$

$$\Sigma = \sigma(H_\omega)$$

$$\Sigma_{pp} = \sigma_{pp}(H_\omega),\ \Sigma_{ac} = \sigma_{ac}(H_\omega),\ \Sigma_{sc} = \sigma_{sc}(H_\omega).$$

Theorem (Kunz-Souillard'80)

Let $H_\omega = -\Delta + V_\omega$ be the Anderson model on $\ell^2(\mathbb{Z}^d)$. Then

$$(*) \qquad \sigma(H_\omega) = \sigma(-\Delta) + \operatorname{supp}\mu \quad \textit{a.s.}$$

Summary

We saw that the Anderson model H_ω in $\ell^2(\mathbb{Z}^d)$ is ergodic. That is, there exists an ergodic group of transformations $(\tau_\gamma)_{\gamma\in\Gamma}$ acting on Ω associated to a family of unitary operators $(U_\gamma)_{\gamma\in\Gamma}$ on $\mathcal{H}$ s.t.

$$H_{\tau_\gamma(\omega)} = U_\gamma H_\omega U_\gamma^* \quad \text{for all}\, \gamma \in \Gamma.$$

- ergodicity $\Rightarrow$ the spectrum of H_ω is deterministic.
  That is, there exists $\Sigma \subset \mathbb{R}$, such that

$$\sigma(H_\omega) = \Sigma \text{ for } \mathbb{P}\text{-a.e.}\, \omega \in \Omega.$$

- ergodicity $\Rightarrow$ the pp/sc/ac spectrum of H_ω is deterministic.
- For H_ω in $\ell^2(\mathbb{Z}^d)$, we can compute the exact set in $\mathbb{R}$ which corresponds to the deterministic spectrum.

- ergodicity $\Rightarrow$ existence of Integrated Density of States.
  Moreover, this function does not depend on $\omega \in \Omega$.
- The IDS gives another way to prove that the spectrum is deterministic.
- In some cases, the IDS gives also information on the localization region !

Reference

- W. Kirsch, *An invitation to Random Schrödinger Operators*, in Random Schrödinger Operators, Panoramas et Syntheses Vol. 25, 2008 (SMF).

The proof of localization

From Fractional Moment Method to localization

What is known

Consider the operator $H_\omega = -\Delta + \lambda V_\omega$, with $\lambda > 0$ acting on $\ell^2(\mathbb{Z}^d)$.

- Recall that H_ω exhibits *dynamical localization* (DL) in I if there exist constants $C < \infty$ and $c > 0$ such that for all $x, y \in \mathbb{Z}^d$,

$$(DL) \qquad \mathbb{E}\left(\sup_{t\in\mathbb{R}} |\langle \delta_y, e^{-itH_\omega}\chi_I(H_\omega)\delta_x\rangle|\right) \leq Ce^{-c|x-y|}$$

Theorem (Localization in $d = 1$)

For any $\lambda > 0$ H_ω exhibits localization throughout its spectrum a.s.

Theorem (Localization $d > 1$)

i. *for $\lambda > 0$ large enough, H_ω exhibits localization throughout its spectrum a.s.*
ii. *for fixed λ, H_ω exhibits localization in intervals I at spectral edges a.s.*

Absence of transport

- Recall that H_ω exhibits *dynamical localization* (DL) in I if there exist constants $C < \infty$ and $c > 0$ such that for all $x, y \in \mathbb{Z}^d$,

$$(DL) \qquad \mathbb{E}\left(\sup_{t\in\mathbb{R}} \left|\langle \delta_y, e^{-itH_\omega}\chi_I(H_\omega)\delta_x\rangle\right|\right) \leq Ce^{-c|x-y|}$$

Theorem (DL implies absence of transport)

If (DL) holds in $J \subset \mathbb{R}$, then for $\varphi \in \ell^2(\mathbb{Z}^d)$ with compact support we have

weighted space → $|X|^{p/2}$; time evolution → e^{-itH_ω}; restriction in energy → $\chi_J(H_\omega)$

$$\sup_t \left\| |X|^{p/2} e^{-itH_\omega} \chi_J(H_\omega)\varphi \right\| < \infty,$$

for every $p \geq 0$, with probability one.

Proof of theorem (DL implies absence of transport)

Recall that $|X|\,\varphi(n) = |n|\,\varphi(n)$ for $\varphi \in \ell^2(\mathbb{Z}^d)$. Take $\varphi \in \ell^2_c(\mathbb{Z}^d)$, that is, for some $R > 0$, $\varphi(n) = 0$ for $|n| > R$, and $\|\varphi\| = 1$. Then, using the expression

$$\|x\|^2 = \sum_j |\langle \delta_j, x\rangle|^2$$

$$\begin{aligned}
\left\| |X|^p e^{-itH_\omega}\chi_I(H_\omega)\varphi \right\|^2 &= \sum_{j\in\mathbb{Z}^d} \left|\langle \delta_j, |X|^p e^{-itH_\omega}\chi_I(H_\omega)\varphi\rangle\right|^2 \\
&\le \sum_j |j|^{2p} \left|\langle \delta_j, e^{-itH_\omega}\chi_I(H_\omega)\varphi\rangle\right|^2 \\
&\le \sum_j |j|^{2p} \left|\langle \delta_j, e^{-itH_\omega}\chi_I(H_\omega)\varphi\rangle\right| \|\varphi\| \\
&\le \sum_j |j|^{2p} \left|\langle \delta_j, e^{-itH_\omega}\chi_I(H_\omega)\left(\sum_{|k|\le R}\langle \varphi, \delta_k\rangle \delta_k\right)\rangle\right| \\
&\le \sum_j \sum_{|k|\le R} |j|^{2p} \left|\langle \delta_j, e^{-itH_\omega}\chi_I(H_\omega)\delta_k\rangle\right|
\end{aligned}$$

$$\left\| |X|^p e^{-itH_\omega} \chi_I(H_\omega) \varphi \right\|^2 \leq \sum_j \sum_{|k| \leq R} |j|^{2p} \left| \langle \delta_j, e^{-itH_\omega} \chi_I(H_\omega) \delta_k \rangle \right|$$

Taking the expectation $\mathbb{E}$ in both sides, we get

$$\begin{aligned}
\mathbb{E}\left(\sup_t \left\| |X|^p e^{-itH_\omega} \chi_I(H_\omega) \varphi \right\|^2 \right) \\
&\leq \sum_j \sum_{|k| \leq R} |j|^{2p} \, \mathbb{E}\left(\sup_t \left| \langle \delta_j, e^{-itH_\omega} \chi_I(H_\omega) \delta_k \rangle \right| \right) \\
&\leq \sum_j \sum_{|k| \leq R} |j|^{2p} \, C e^{-c|j-k|} \qquad (DL) \\
&< \infty
\end{aligned}$$

Finally, if $\mathbb{E}(f) < \infty$, then $f < \infty$ a.s. Therefore, for any $p \geq 0$,

$$\sup_t \left\| |X|^p e^{-itH_\omega} \chi_I(H_\omega) \varphi \right\|^2 < \infty \quad a.s.$$

□

Pure point spectrum

Recall that

- We say that H_ω exhibits *dynamical localization* (DL) in I if there exist constants $C < \infty$ and $c > 0$ such that for all $x, y \in \mathbb{Z}^d$,

$$(DL) \qquad \mathbb{E}\left(\sup_{t\in\mathbb{R}} |\langle \delta_y, e^{-itH_\omega}\chi_I(H_\omega)\delta_x\rangle|\right) \le Ce^{-c|x-y|}$$

Theorem (DL implies pure point spectrum)

If (DL) holds in an interval I, then H_ω has pure point spectrum in I with probability one.

The proof relies on the RAGE Theorem.

Theorem (Ruelle-Amrein-Georgescu-Enss)

Let H be a s.a. operator on $\ell^2(\mathbb{Z}^d)$, let P_c and P_{pp} be the orthogonal projections onto $\mathcal{H}_c$ and $\mathcal{H}_{pp}$, resp. Let Λ_L be a cube of side L around the origin. Then, for any $\varphi \in \ell^2(\mathbb{Z}^d)$,

$$\|P_c\varphi\|^2 = \lim_{L\to\infty} \lim_{T\to\infty} \frac{1}{T}\int_0^T \left(\sum_{x\notin\Lambda_L} |e^{-itH}\varphi(x)|^2 \right) dt$$

$$\|P_{pp}\varphi\|^2 = \lim_{L\to\infty} \lim_{T\to\infty} \frac{1}{T}\int_0^T \left(\sum_{x\in\Lambda_L} |e^{-itH}\varphi(x)|^2 \right) dt$$

Proof : e.g. see Kirsch's notes.

Take $\varphi \in \ell_c(\mathbb{Z}^d)$, that is, for some $R > 0$, $\varphi(n) = 0$ for $|n| > R$. From RAGE Theorem we have that

$$\|P_c(H_\omega)\chi_I(H_\omega)\varphi\|^2 = \lim_{L\to\infty}\lim_{T\to\infty}\frac{1}{T}\int_0^T \left(\sum_{x\notin\Lambda_L}|e^{-itH}\chi_I(H_\omega)\varphi(x)|^2\right)dt$$

Note that

$$\sum_{x\notin\Lambda_L}|e^{-itH}\chi_I(H_\omega)\varphi(x)|^2 = \left\|\chi_{\Lambda_L^c}e^{-itH}\chi_I(H_\omega)\varphi\right\|^2 = \left\|\chi_{\Lambda_L^c}e^{-itH}\chi_I(H_\omega)\chi_{\Lambda_R}\varphi\right\|^2$$
$$\leq \left\|\chi_{\Lambda_L^c}e^{-itH}\chi_I(H_\omega)\chi_{\Lambda_R}\right\|^2\|\varphi\|^2$$
$$\leq \sum_{|x|\geq L}\sum_{|k|\leq R}\left|\langle\delta_x, e^{-itH}\chi_I(H_\omega)\delta_k\rangle\right|\|\varphi\|^2$$

Taking the expectation $\mathbb{E}$ in both sides, and using Fatou's lemma and Fubini, yields

$$\mathbb{E}(\|P_c(H_\omega)\chi_I(H_\omega)\varphi\|^2)$$
$$\leq \lim_{L\to\infty}\lim_{T\to\infty}\frac{1}{T}\int_0^T\sum_{|x|\geq L}\sum_{|k|\leq R}\|\varphi\|^2\,\mathbb{E}\left(\left|\langle\delta_x, e^{-itH}\chi_I(H_\omega)\delta_k\rangle\right|\right)$$

$$\mathbb{E}(\|P_c(H_\omega)\chi_I(H_\omega)\varphi\|^2)$$
$$\leq \lim_{L\to\infty} \lim_{T\to\infty} \frac{1}{T}\int_0^T \sum_{|x|\geq L} \sum_{|k|\leq R} \|\varphi\|^2 \mathbb{E}\left(\left|\langle \delta_x, e^{-itH}\chi_I(H_\omega)\delta_k\rangle\right|\right)$$

Note that by hypothesis (dynamical localization),

$$\mathbb{E}\left(\left|\langle \delta_x, e^{-itH}\chi_I(H_\omega)\delta_k\rangle\right|\right) \leq Ce^{-c|x-k|}$$

uniformly in t, then

$$\mathbb{E}(\|P_c(H_\omega)\chi_I(H_\omega)\varphi\|^2) \leq C\|\varphi\|^2 \lim_{L\to\infty} \sum_{|x|\geq L} \sum_{|k|\leq R} e^{-c|x-k|}$$

Since the sum in the r.h.s is convergent, the limit when $R \to \infty$ is 0. Then

$$\mathbb{E}(\|P_c(H_\omega)\chi_I(H_\omega)\varphi\|^2) = 0$$

implies $P_c(H_\omega)\chi_I(H_\omega)\varphi = 0$ for almost every $\omega \in \Omega$ and $\varphi \in \ell_c(\mathbb{Z}^d)$. Since $\ell_c(\mathbb{Z}^d)$ is dense in $\ell^2(\mathbb{Z}^d)$, the result follows. □

Alternative proof (absence of transport implies pure point spectrum).

Take $\varphi \in \ell_c(\mathbb{Z}^d)$, that is, for some $R > 0$, $\varphi(n) = 0$ for $|n| > R$. From RAGE Theorem we have that

$$\|P_c(H_\omega)\chi_I(H_\omega)\varphi\|^2 = \lim_{L\to\infty} \lim_{T\to\infty} \frac{1}{T}\int_0^T \left(\sum_{x\notin\Lambda_L} |e^{-itH}\chi_I(H_\omega)\varphi(x)|^2 \right) dt$$

Note that

$$\sum_{x\notin\Lambda_L} |e^{-itH}\chi_I(H_\omega)\varphi(x)|^2 \le \sum_{x\notin\Lambda_L} \frac{1}{|x|^{2p}} |\,|X|^p e^{-itH}\chi_I(H_\omega)\varphi(x)|^2$$

$$\le \|\,|X|^p e^{-itH}\chi_I(H_\omega)\varphi(x)\|^2 \sum_{x\notin\Lambda_L} \frac{1}{|x|^{2p}}$$

Therefore,

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \|\,|X|^p e^{-itH}\chi_I(H_\omega)\varphi(x)\|^2 dt < C$$

Which leaves

$$\|P_c(H_\omega)\chi_I(H_\omega)\varphi\|^2 \le C \lim_{L\to\infty} \sum_{x\notin\Lambda_L} \frac{1}{|x|^{2p}} = 0$$

□

How to prove localization ?

- Study decay of the resolvent.

Theorem (Fractional Moment Method (FMM))

If for a given $I \subset \sigma(H_\omega)$ a.s., the following holds : there exists $s \in (0,1)$ and $0 < c, C < \infty$ such that

$$(\star) \qquad \mathbb{E}\left(\left|\langle \delta_x, (H_{\omega,\lambda} - (E + i\varepsilon))^{-1} \delta_y \rangle\right|^s\right) \leq Ce^{-c\|x-y\|}$$

uniformly in $E \in I$, $\varepsilon > 0$ and $x, y \in \mathbb{Z}^d$, then the operator H_ω exhibits dynamical localization in I.

- Relate resolvent decay to dynamical localization.

$$\mathbb{E}\left(\sup_{f:\mathbb{R}\to\mathbb{C},|f|\le 1}|\langle\delta_x, f(H)\chi_I(H)\delta_y\rangle|\right)$$
$$\le C\liminf_{|\varepsilon|\to 0}\int_I\sum_z \mathbb{E}\left(\left|G_{\omega,\lambda}(x,z;E+i\varepsilon)\right|^s\right)^{1/2}\mathbb{E}\left(\left|G_{\omega,\lambda}(z,y;E-i\varepsilon)\right|^s\right)^{1/2}dE.$$

Proof : Graf'94, based on Stone's theorem, see also Stolz's notes.

In particular, this gives dynamical localization.

$$(DL)\qquad \mathbb{E}\left(\sup_{t\in\mathbb{R}}|\langle\delta_x, e^{-itH_{\omega,\lambda}}\chi_I(H_{\omega,\lambda})\delta_y\rangle|\right)\le Ce^{-c|x-y|}.$$

- The Simon-Wolff Criterion : works if the probability distribution of the random variables is absolutely continuous. Then, if for Lebesgue-a.e. $E\in I$ and $\mathbb{P}$-a.e. ω

$$\lim_{\varepsilon\to 0}\sum_{y\in\mathbb{Z}^d}\left|\langle\delta_y,(H_\omega-(E+i\varepsilon))^{-1}\delta_x\rangle\right|^2<\infty,$$

then the spectral measure associated with δ_x is pure point in I for $\mathbb{P}$-a.e. ω.

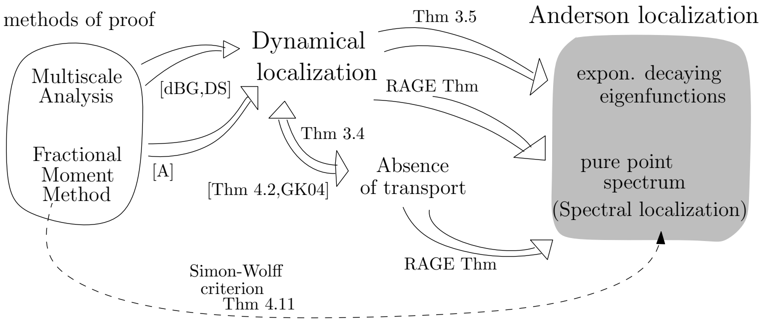


FIGURE – Summary taken from [RM'17] of relation between localization and methods

Fractional Moment Method

Proof of localization at high disorder

Reference :
We follow closely Section 4 in G. Stolz's notes *An introduction to the mathematics of Anderson localization*, Contemporary Mathematics 551, 2010.

In the rest of this lecture, we will focus on showing

Theorem

Let $s \in (0,1)$. Then there exists $\lambda_0 > 0$ such that for $\lambda \geq \lambda_0$, there are constants $0 < c$, $C < \infty$ such that

$$(*) \qquad \mathbb{E}\left(\left|\langle \delta_x, (H_{\omega,\lambda} - z)^{-1}\delta_y\rangle\right|^s\right) \leq Ce^{-c\|x-y\|}$$

uniformly in $x, y \in \mathbb{Z}^d$ and $z \in \mathbb{C} \setminus \mathbb{R}$.

We assume the random variables ω_n have an absolutely continuous probability distribution, with a continuous density, i.e., there exists $\rho \in \mathcal{C}(\mathbb{R})$ s.t.

$$d\mu(x) = \rho(x)dx$$

The proof relies on two results :

- An a priori bound on the fractional moment of the resolvent :

$$\mathbb{E}\left(\left|\langle \delta_x, (H_{\omega,\lambda} - z)^{-1}\delta_y\rangle\right|^s\right) \le C(s,\lambda,\rho).$$

- A decoupling lemma : for ρ there exists a constant $C < \infty$ s.t., uniformly in α and $\beta \in \mathbb{C}$,

$$\int \frac{1}{|v-\beta|^s}\rho(v)dv \le C \int \frac{|v-\alpha|^s}{|v-\beta|^s}\rho(v)dv$$

The *a priori* bound

Since the random variables ω_n have a probability density ρ, compactly supported and bounded, we can write

$$\mathbb{E}(\cdot) := \int_{\Omega} (\cdot) d\mathbb{P} = \int_{\mathbb{A}} \dots \int_{\mathbb{A}} (\cdot) \dots g(\omega_n) d\omega_n \dots$$

Lemma (A priori bound)

There exists a constant $C = (s, \rho) < \infty$ *such that*

$$\mathbb{E}\left(\left| \langle \delta_x, (H_{\omega,\lambda} - z)^{-1} \delta_y \rangle \right|^s \right) \leq \frac{C(s,\rho)}{\lambda^s},$$

for all $x, y \in \mathbb{Z}^d$ *and* $\lambda > 0$.

Proof : we will start by showing that

$$\mathbb{E}_{x,y}\left(\left| \langle \delta_x, (H_{\omega,\lambda} - z)^{-1} \delta_y \rangle \right|^s \right) \leq \frac{C(s,\rho)}{\lambda^s}.$$

We will use the conditional expectation with $(\omega_n)_{n \neq x,y}$ fixed.

$$\mathbb{E}_{x,y}(\cdot) = \int_{\mathbb{A}} \int_{\mathbb{A}} (\cdot) \rho(\omega_x)\, \rho(\omega_y)\, d\omega_x\, d\omega_y.$$

Note that if we are able to show

$$\mathbb{E}_{x,y}\left(\left| \langle \delta_x, (H_{\omega,\lambda} - z)^{-1} \delta_y \rangle \right|^s \right) \leq \frac{C(s,\rho)}{\lambda^s},$$

the r.h.s does not depend on $(\omega_n)_{n \notin \{x,y\}}$ anymore. We can then take the $\mathbb{E}$ with respect to the rest of the r.v. and obtain

$$\mathbb{E}\left(\left| \langle \delta_x, (H_{\omega,\lambda} - z)^{-1} \delta_y \rangle \right|^s \right) \leq \frac{C(s,\rho)}{\lambda^s},$$

which is the desired result.

Proof of the a priori bound

Goal : to obtain an upper bound for

$$\mathbb{E}_{x,y}\left(\left|\langle\delta_x,(H_{\omega,\lambda}-z)^{-1}\delta_y\rangle\right|^s\right),\ x,y\in\mathbb{Z}^d.$$

We split the proof in two cases : i) when $x=y$ and ii) when $x\neq y$.

i) Case $x=y$ (rank-one perturbation)
Recall that

$$H_{\omega,\lambda}=-\Delta+\lambda\sum_{n\in\mathbb{Z}^d}\omega_n P_n,\quad P_n:=\langle\delta_n,\cdot\rangle\delta_n.$$

Write $\omega=(\hat{\omega},\omega_x)$, where $\hat{\omega}=(\omega_n)_{n\neq x}$. Then

$$H_{\omega,\lambda}=H_{\hat{\omega},\lambda}+\lambda\omega_x P_x$$

Using the resolvent identity, we get

$$\left(H_{\omega,\lambda}-z\right)^{-1}=\left(H_{\hat{\omega},\lambda}-z\right)^{-1}-\lambda\omega_x\left(H_{\hat{\omega},\lambda}-z\right)^{-1}P_x\left(H_{\omega,\lambda}-z\right)^{-1}$$

$$\left(H_{\omega,\lambda}-z\right)^{-1}=\left(H_{\hat{\omega},\lambda}-z\right)^{-1}-\lambda\omega_x\left(H_{\hat{\omega},\lambda}-z\right)^{-1}P_x\left(H_{\omega,\lambda}-z\right)^{-1}$$

Now we take matrix-elements i.e. compute $\langle\delta_x,\cdot\rangle$ in both sides :

$$\begin{aligned}\langle\delta_x,\left(H_{\omega,\lambda}-z\right)^{-1}\delta_x\rangle &= \langle\delta_x,\left(H_{\hat{\omega},\lambda}-z\right)^{-1}\delta_x\rangle \\ &\quad -\lambda\omega_x\langle\delta_x,\left(H_{\hat{\omega},\lambda}-z\right)^{-1}\delta_x\rangle\langle\delta_x,\left(H_{\omega,\lambda}-z\right)^{-1}\delta_x\rangle\end{aligned}$$

Using the notation $G_{\omega,\lambda}(x,y;z):=\langle\delta_x,\left(H_{\hat{\omega},\lambda}-z\right)^{-1}\delta_x\rangle$, we get

$$G_{\omega,\lambda}(x,x;z)=G_{\hat{\omega},\lambda}(x,x;z)-\lambda\omega_x G_{\hat{\omega},\lambda}(x,x;z)G_{\omega,\lambda}(x,x;z).$$

If we write $\alpha=\alpha(\hat{\omega},x,z):=\left(G_{\hat{\omega},\lambda}(x,x;z)\right)^{-1}$, then

$$G_{\omega,\lambda}(x,x;z)=\frac{1}{\alpha+\lambda\omega_x}.$$

Here, α is well-defined, because $\frac{\operatorname{Im} G_{\hat{\omega},\lambda}(x,x;z)}{\operatorname{Im} z}>0$.

$$G_{\omega,\lambda}(x,x;z) = \frac{1}{\alpha + \lambda\omega_x},$$

where $\alpha \in \mathbb{C}$ and does not depend on ω_x !
Suppose $\mathrm{supp}\rho \subset [-M, M]$. Then

$$\begin{aligned}\mathbb{E}_x\left(\left|G_{\omega,\lambda}(x,x;z)\right|^s\right) &= \int_{-M}^{M} \frac{1}{\left|\alpha + \lambda\omega_x\right|^s}\rho(\omega_x)\, d\omega_x \\ &\leq \frac{\|\rho\|_\infty}{\lambda^s}\int_{-M}^{M} \frac{1}{\left|\alpha\lambda^{-1} + \omega_x\right|^s} d\omega_x.\end{aligned}$$

The r.h.s is integrable, independent of α and λ. Therefore,

$$\mathbb{E}_x\left(\left|G_{\omega,\lambda}(x,x;z)\right|^s\right) \leq \frac{C(\rho,s)}{\lambda^s}.$$

which is the desired bound for $x = y$.

ii) Case $x \neq y$ (rank-two perturbation)
Recall that

$$H_{\omega,\lambda} = -\Delta + \sum_{n \in \mathbb{Z}^d} \omega_n P_n, \quad P_n := \langle \delta_n, \cdot \rangle \delta_n.$$

Write $\omega = (\hat{\omega}, \omega_x, \omega_y)$, with $\hat{\omega} = (\omega_n)_{n \notin \{x,y\}}$, then

$$H_{\omega,\lambda} = H_{\hat{\omega},\lambda} + \lambda\omega_x P_x + \lambda\omega_y P_y.$$

Writing $P = P_x + P_y$ and using the resolvent identity, we get

$$\left(H_{\omega,\lambda} - z\right)^{-1} = \left(H_{\hat{\omega},\lambda} - z\right)^{-1} - \left(H_{\omega,\lambda} - z\right)^{-1} \left(\lambda\omega_x P_x + \lambda\omega_y P_y\right) \left(H_{\hat{\omega},\lambda} - z\right)^{-1}$$

Now, we want to determine the matrix-elements (omit z for convenience)

$$\begin{pmatrix} G_{\omega,\lambda}(x,x) & G_{\omega,\lambda}(x,y) \\ G_{\omega,\lambda}(y,x) & G_{\omega,\lambda}(y,y) \end{pmatrix}$$

in terms of

$$\begin{pmatrix} G_{\hat{\omega},\lambda}(x,x) & G_{\hat{\omega},\lambda}(x,y) \\ G_{\hat{\omega},\lambda}(y,x) & G_{\hat{\omega},\lambda}(y,y) \end{pmatrix}$$

Using

$$\left(H_{\omega,\lambda}-z\right)^{-1}=\left(H_{\hat{\omega},\lambda}-z\right)^{-1}-\left(H_{\omega,\lambda}-z\right)^{-1}\left(\lambda\omega_x P_x+\lambda\omega_y P_y\right)\left(H_{\hat{\omega},\lambda}-z\right)^{-1}.$$

we can compute each matrix element, for ex.

$$G_{\omega,\lambda}(x,x)=G_{\hat{\omega},\lambda}(x,x)-\lambda\omega_x G_{\omega,\lambda}(x,x)G_{\hat{\omega},\lambda}(x,x)-\lambda\omega_y G_{\omega,\lambda}(x,y)G_{\hat{\omega},\lambda}(y,x).$$

After some computations... we get

$$\begin{pmatrix} G_{\omega,\lambda}(x,x) & G_{\omega,\lambda}(x,y) \\ G_{\omega,\lambda}(y,x) & G_{\omega,\lambda}(y,y) \end{pmatrix}=\left[\begin{pmatrix} G_{\hat{\omega},\lambda}(x,x) & G_{\hat{\omega},\lambda}(x,y) \\ G_{\hat{\omega},\lambda}(y,x) & G_{\hat{\omega},\lambda}(y,y) \end{pmatrix}^{-1}+\lambda\begin{pmatrix} \omega_x & 0 \\ 0 & \omega_y \end{pmatrix}\right]^{-1}$$

$$=:\left[G_{\hat{\omega}}^{-1}+\lambda\begin{pmatrix} \omega_x & 0 \\ 0 & \omega_y \end{pmatrix}\right]^{-1}$$

Since $G_{\omega,\lambda}(x,y;z)$ is one element of the matrix, we can bound it by the norm of the matrix

$$\mathbb{E}\left(\left|G_{\omega,\lambda}(x,y;z)\right|^s\right)\leq\mathbb{E}_{x,y}\left(\left\|\left[G_{\hat{\omega}}^{-1}+\lambda\begin{pmatrix} \omega_x & 0 \\ 0 & \omega_y \end{pmatrix}\right]^{-1}\right\|^s\right).$$

$$\mathbb{E}\left(\left|G_{\omega,\lambda}(x,y;z)\right|^s\right) \leq \frac{1}{\lambda^s}\mathbb{E}_{x,y}\left(\left\|\left[\frac{1}{\lambda}G_{\hat{\omega}}^{-1} + \begin{pmatrix} \omega_x & 0 \\ 0 & \omega_y \end{pmatrix}\right]^{-1}\right\|^s\right)$$

$$= \frac{1}{\lambda^s}\int\int\left\|\left[\frac{1}{\lambda}G_{\hat{\omega}}^{-1} + \begin{pmatrix} \omega_x & 0 \\ 0 & \omega_y \end{pmatrix}\right]^{-1}\right\|^s \rho(\omega_x)\,\rho(\omega_y)\,d\omega_x\,d\omega_y$$

$$\leq \frac{\|\rho\|_\infty^2}{\lambda^s}\int_{-M}^{M}\int_{-M}^{M}\left\|\left[\frac{1}{\lambda}G_{\hat{\omega}}^{-1} + \begin{pmatrix} \omega_x & 0 \\ 0 & \omega_y \end{pmatrix}\right]^{-1}\right\|^s d\omega_x\,d\omega_y,$$

Now, we would like to decouple the matrix with elements ω_x, ω_y, and isolate each term. For this, we do a change of variables

$$u = \frac{\omega_x + \omega_y}{2}, \quad v = \frac{\omega_x - \omega_y}{2},$$

and get

$$\mathbb{E}\left(\left|G_{\omega,\lambda}(x,y;z)\right|^s\right) \leq \frac{2\,\|\rho\|_\infty^2}{\lambda^s}\int_{-M}^{M}\int_{-M}^{M}\left\|\left[\frac{1}{\lambda}G_{\hat{\omega}}^{-1} + \begin{pmatrix} -v & 0 \\ 0 & v \end{pmatrix} + u\mathbb{I}_{2x2}\right]^{-1}\right\|^s du\,dv$$

$$\mathbb{E}\left(\left|G_{\omega,\lambda}(x,y;z)\right|^s\right) \leq \frac{2\,\|\rho\|_\infty^2}{\lambda^s}\int_{-M}^{M}\int_{-M}^{M}\left\|\left[\frac{1}{\lambda}G_{\hat{\omega}}^{-1} + \begin{pmatrix} -v & 0 \\ 0 & v \end{pmatrix} + u\mathbb{I}_{2x2}\right]^{-1}\right\|^s du\,dv$$

Note that the matrix

$$\frac{1}{\lambda}G_{\hat{\omega}}^{-1} + \begin{pmatrix} -v & 0 \\ 0 & v \end{pmatrix}$$

has either positive or negative imaginary part.

Therefore we can use the following result :

Lemma : *For all* 2×2 *matrices* A *such that either* $\mathrm{Im}A \geq 0$ *or* $\mathrm{Im}A \leq 0$*, one has*

$$\int_{-M}^{M}\left\|(A+u\mathbb{I})^{-1}\right\|^s du \leq C(M,s).$$

For a proof, see G. Stolz's notes.

We obtain

$$\mathbb{E}\left(\left|G_{\omega,\lambda}(x,y;z)\right|^s\right) \leq 4M\,\|\rho\|_\infty^2\,C(M,s)\frac{1}{\lambda^s}$$

□

Remarks

In the last proof we obtained the following

$$\begin{pmatrix} G_{\omega,\lambda}(x,x) & G_{\omega,\lambda}(x,y) \\ G_{\omega,\lambda}(y,x) & G_{\omega,\lambda}(y,y) \end{pmatrix} = \left[\begin{pmatrix} G_{\hat{\omega},\lambda}(x,x) & G_{\hat{\omega},\lambda}(x,y) \\ G_{\hat{\omega},\lambda}(y,x) & G_{\hat{\omega},\lambda}(y,y) \end{pmatrix}^{-1} + \lambda \begin{pmatrix} \omega_x & 0 \\ 0 & \omega_y \end{pmatrix} \right]^{-1}$$

This is a special case of a more general result, called *the Krein formula*.

Theorem (Krein formula)

Let H be a self-adjoint operator on some Hilbert space $\mathcal{H}$. If

$$H = H_0 + W,$$

with W a finite rank operator satisfying

$$W = PWP$$

for some finite-dimensional orthogonal projection P, then, for z with $\mathrm{Im} z \neq 0$, we have

$$\left[P(H-z)^{-1}P\right] = \left[W + \left[P(H_0-z)^{-1}P\right]^{-1}\right]^{-1}$$

where the inverse is taken on the restriction to the range of P.

Let us recall that we want to prove the following

Theorem

Let $s \in (0,1)$. Then there exists $\lambda_0 > 0$ such that for $\lambda \geq \lambda_0$, there are constants $0 < c$, $C < \infty$ such that

$$(*) \qquad \mathbb{E}\left(\left|\langle \delta_x, (H_{\omega,\lambda} - z)^{-1}\delta_y\rangle\right|^s\right) \leq Ce^{-c\|x-y\|}$$

uniformly in $x, y \in \mathbb{Z}^d$ and $z \in \mathbb{C} \setminus \mathbb{R}$.

Ingredients of the proof :

- The a priori bound on the fractional moment of the resolvent :

$$\mathbb{E}\left(\left|\langle \delta_x, (H_{\omega,\lambda} - z)^{-1}\delta_y\rangle\right|^s\right) \leq C(s,\lambda,\rho).$$

- A decoupling lemma : for ρ there exists a constant $C' < \infty$ s.t., uniformly in α and $\beta \in \mathbb{C}$,

$$\int \frac{1}{|v-\beta|^s}\rho(v)dv \leq C \int \frac{|v-\alpha|^s}{|v-\beta|^s}\rho(v)dv$$

Proof of Theorem

Suppose $x \neq y$. Then $\langle \delta_x, \delta_y \rangle = 0$ and

$$\begin{aligned}
\langle \delta_x, \delta_y \rangle &= \langle \delta_x, (H_{\omega,\lambda} - z)^{-1} (H_{\omega,\lambda} - z)\, \delta_y \rangle \\
&= \left\langle \delta_x, (H_{\omega,\lambda} - z)^{-1} (-\Delta\delta_y - (V_\omega - z)\delta_y) \right\rangle \\
&= \left\langle \delta_x, (H_{\omega,\lambda} - z)^{-1} \left(- \sum_{u \sim y} \delta_u - (\lambda\omega_y - z)\delta_y \right) \right\rangle \\
&= \left\langle \delta_x, (H_{\omega,\lambda} - z)^{-1} \left(- \sum_{u \sim y} \delta_u \right) \right\rangle + (\lambda\omega_y - z) \left\langle \delta_x, (H_{\omega,\lambda} - z)^{-1} \delta_y \right\rangle \\
&= - \sum_{u \sim y} G_{\omega,\lambda}(x, u; z) + (\lambda\omega_y - z) G_{\omega,\lambda}(x, y; z).
\end{aligned}$$

One can compute that

$$G_{\omega,\lambda}(x, y; z) = \frac{a}{\lambda\omega_y - b},$$

where a and b do not depend on ω_y.

$$\mathbb{E}\left(\left|G_{\omega,\lambda}(x,y;z)\right|^s\right) = \frac{1}{\lambda^s}\mathbb{E}\left(\frac{|a|^s}{\left|\omega_y - \frac{b}{\lambda}\right|^s}\right)$$

$$\leq \frac{C'}{\lambda^s}\mathbb{E}\left(\frac{\left|\omega_y - \frac{z}{\lambda}\right|^s |a|^s}{\left|\omega_y - \frac{b}{\lambda}\right|^s}\right) \qquad \text{decoupling lemma}$$

$$= \frac{C'}{\lambda^s}\mathbb{E}\left(|\lambda\omega_y - z|^s \left|G_{\omega,\lambda}(x,y;z)\right|^s\right)$$

where we used that

$$G_{\omega,\lambda}(x,y;z) = \frac{a}{\lambda\omega_y - b}.$$

Recall that we had shown that

$$(\lambda\omega_y - z)G_{\omega,\lambda}(x,y;z) = \sum_{u\sim y} G_{\omega,\lambda}(x,u;z).$$

Therefore, using that $(\sum_n |a_n|)^s \leq \sum_n |a_n|^s$, we get

$$\mathbb{E}\left(\left|G_{\omega,\lambda}(x,y;z)\right|^s\right) \leq \frac{C'}{\lambda^s}\sum_{u\sim y}\mathbb{E}\left(\left|G_{\omega,\lambda}(x,u;z)\right|^s\right).$$

$$\mathbb{E}\left(\left|G_{\omega,\lambda}(x,y;z)\right|^s\right) \leq \frac{C'}{\lambda^s}\sum_{u\sim y}\mathbb{E}\left(\left|G_{\omega,\lambda}(x,u;z)\right|^s\right).$$

If none of the points u is equal to x, we can iterate this argument.

$$\begin{aligned}\mathbb{E}\left(\left|G_{\omega,\lambda}(x,y;z)\right|^s\right) &\leq \frac{C'}{\lambda^s}\sum_{u\sim y}\mathbb{E}\left(\left|G_{\omega,\lambda}(x,u;z)\right|^s\right)\\ &\leq \frac{C'}{\lambda^s}\,(\sharp\text{of neighbors})\max_{\substack{u\\ u\sim y}}\mathbb{E}\left(\left|G_{\omega,\lambda}(x,u;z)\right|^s\right)\\ &\leq \left(\frac{C'}{\lambda^s}\right)^2(\sharp\text{of neighbors})\sum_{u'\sim u}\mathbb{E}\left(\left|G_{\omega,\lambda}(x,u';z)\right|^s\right)\end{aligned}$$

iterating this argument, at each step we get a factor

$$\left(\frac{C'}{\lambda^s}\right)(\sharp\text{of neighbors})$$

We can iterate this argument at most $\|x-y\|$ times,

$$\mathbb{E}\left(\left|G_{\omega,\lambda}(x,y;z)\right|^s\right) \leq \left(\left(\frac{C'}{\lambda^s}\right)^2 (\sharp\text{of neighbors})\right)^{\|x-y\|} \sup_{u\in\mathbb{Z}^d} \mathbb{E}\left(\left|G_{\omega,\lambda}(x,u;z)\right|^s\right)$$

We can bound the r.h.s using the a priori bound and get

$$\mathbb{E}\left(\left|G_{\omega,\lambda}(x,y;z)\right|^s\right) \leq \frac{C(\rho,s)}{\lambda^s} \left(\left(\frac{C'}{\lambda^s}\right)^2 (\sharp\text{of neighbors})\right)^{\|x-y\|}$$

Finally, we take λ large enough such that

$$\left(\left(\frac{C'}{\lambda^s}\right)^2 2d\right) < 1.$$

Then, we have

$$\mathbb{E}\left(\left|G_{\omega,\lambda}(x,y;z)\right|^s\right) \leq \frac{C(\rho,s)}{\lambda^s} e^{-C(C',\lambda,s,d)\|x-y\|}.$$

□

We have shown

Theorem

Let $s \in (0,1)$. Then there exists $\lambda_0 > 0$ such that for $\lambda \geq \lambda_0$, there are constants $0 < c$, $C < \infty$ such that

$$(*) \qquad \mathbb{E}\left(\left| \langle \delta_x, (H_{\omega,\lambda} - z)^{-1} \delta_y \rangle \right|^s \right) \leq Ce^{-c\|x-y\|}$$

uniformly in $x, y \in \mathbb{Z}^d$ and $z \in \mathbb{C} \setminus \mathbb{R}$.

With this result, we can prove dynamical localization, and pure point spectrum. For a proof of dynamical localization, see Section 5 in G. Stolz's notes.

Theorem (The Simon-Wolff Criterion, Simon-Wolff'86)

Let Γ be a countable set of points. Let $H_\omega = -\Delta + V_\omega$ on $\ell^2(\Gamma)$, such that the probability distribution of the random variables, μ, is absolutely continuous. Then, for any Borel set I :

- *If for Lebesgue-a.e. $E \in I$ and $\mathbb{P}$-a.e. ω*

  $$\lim_{\varepsilon \to 0} \sum_{y \in \Gamma} \left| \langle \delta_y, (H_\omega - (E + i\varepsilon))^{-1} \delta_x \rangle \right|^2 < \infty,$$

  then for $\mathbb{P}$-a.e. ω, the spectral measure of H associated to δ_x is pure point in I.

- *If for Lebesgue-a.e. $E \in I$ and $\mathbb{P}$-a.e. ω*

  $$\lim_{\varepsilon \to 0} \sum_{y \in \Gamma} \left| \langle \delta_y, (H_\omega - (E + i\varepsilon))^{-1} \delta_x \rangle \right|^2 = \infty,$$

  then for $\mathbb{P}$-a.e. ω, the spectral measure of H associated to δ_x is continuous in I.

To prove pp spectrum, we would like to use the Simon-Wolff Criterion. Recall our result, which holds for any given $s \in (0,1)$, in the whole spectrum with λ large enough, uniformly on $z = E + i\varepsilon$, $\varepsilon > 0$,

$$\mathbb{E}\left(\left|\langle \delta_x, (H_{\omega,\lambda} - z)^{-1} \delta_y \rangle\right|^s\right) \leq Ce^{-c\|x-y\|}$$

Then

$$\mathbb{E}\left(\sum_y \left|\langle \delta_x, (H_{\omega,\lambda} - z)^{-1} \delta_y \rangle\right|^s\right) \leq \sum_y \mathbb{E}\left(\left|\langle \delta_x, (H_{\omega,\lambda} - z)^{-1} \delta_y \rangle\right|^s\right) < \infty.$$

which implies that

$$\sum_y \left|\langle \delta_x, (H_{\omega,\lambda} - z)^{-1} \delta_y \rangle\right|^s < \infty \quad \text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega.$$

Because the bound is uniform on ε, we an take the limit when $\varepsilon \to 0$.

We use the inequality : *If* $s \in (0,1)$*,*

$$\left(\sum_n |a_n|\right)^s \leq \sum_n |a_n|^s .$$

Take $s = 1/4$,

$$\left(\sum_y \left|\langle \delta_x, (H_{\omega,\lambda} - z)^{-1}\delta_y\rangle\right|^2\right)^{\frac{1}{4}} \leq \sum_y \left|\langle \delta_x, (H_{\omega,\lambda} - z)^{-1}\delta_y\rangle\right|^{\frac{1}{2}} < \infty$$

for $\mathbb{P}$-a.e. $\omega \in \Omega$. Therefore, by the Simon-Wolff Criterion, the spectral measure associated to H_ω and δ_x is pure point in the deterministic spectrum of H_ω, for $\mathbb{P}$-a.e. $\omega \in \Omega$. Since this holds for every δ_x, one can deduce that

$$\sigma(H_\omega) = \sigma_{pp}(H_\omega) \quad \text{for } \mathbb{P}\text{-a.e.}\, \omega \in \Omega.$$

Fractional Moment Method

Proof of localization at the bottom of the spectrum.

Reference :
We follow closely Section 4 in G. Stolz's notes *An introduction to the mathematics of Anderson localization*, Contemporary Mathematics 551, 2010.

Let $H_\omega = -\Delta + \lambda V_\omega$ and fix the disorder λ. Say, $\lambda = 1$. Assume the ω_x are iid with bounded probability density supported in $[0, M]$, $M > 0$. Then

$$\sigma(H_\omega) = [0, 4d] + [0, M] = [0, 4d + M] \quad \text{a.s.}$$

Theorem

For any $s \in (0, 1)$, there exists $\delta > 0$, $C_1 < \infty$ and $C_2 > 0$ such that

$$(**) \qquad \mathbb{E}\left(\left|\langle \delta_x, (H_\omega - (E + i\varepsilon))^{-1} \delta_y \rangle\right|^s\right) \leq C_1 e^{-C_2 \|x-y\|}$$

for all $x, y \in \mathbb{Z}^d$, $E \in [0, \delta]$ and $\varepsilon > 0$.

This implies that H_ω exhibits localization in $I = [0, \delta]$.

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Problem : The a priori bound (Lemma 1) still holds, but we cannot use λ to make the bound as small as we want.

Way out : Restrict to a finite volume. Take a cube of side L, Λ and consider the restriction $H_{\omega,\Lambda}$. Approximate LHS of (**) with

$$\mathbb{E}\left(\left|\langle \delta_x, (H_{\omega,\Lambda} - (E + i\varepsilon))^{-1} \delta_y \rangle\right|^s\right) \quad \text{when} \quad \Lambda \to \mathbb{Z}^d$$

Take a cube Λ of side L and write

$$H_\omega = H_{\omega,\Lambda} \oplus H_{\omega,\Lambda^c} + T_L = H_{\omega,L} + T_L,$$

where

$$\langle \delta_x, H_{\omega,\Lambda} \oplus H_{\omega,\Lambda^c} \delta_y \rangle = \begin{cases} \langle \delta_x, H_{\omega,\Lambda} \delta_y \rangle, & \text{if } x, y \in \Lambda \\ \langle \delta_x, H_{\omega,\Lambda^c} \delta_y \rangle, & \text{if } x, y \in \Lambda^c \\ 0 & \text{otherwise} \end{cases},$$

the boundary operator T_L is given by

$$\langle \delta_x, T_L \delta_y \rangle = \begin{cases} -1, & \text{if } (x, y) \in \partial\Lambda \\ 0 & \text{otherwise} \end{cases}.$$

Apply geometric resolvent identity twice to get

$$G_\omega = G_{\omega,L} - G_{\omega,L} T_L \left(G_{\omega,L+1} - G_\omega T_{L+1} G_{\omega,L+1} \right)$$

For simplicity, consider $x = 0$ and compute decay between points $0, y$. Take L such that $|y| \geq L+2$. Then

$$G_\omega(0,y;z) = \sum_{(u,v)\in\partial\Lambda_L} \sum_{(u',v')\in\partial\Lambda_{L+1}} G_{\omega,L}(0,u;z)G_\omega(u',v;z)G_{\omega,L+1}(v',y;z)$$

Take $|\cdot|^s$, a bit of algebra and then $\mathbb{E}$:

$$\mathbb{E}\left|G_\omega(0,y;z)\right|^s \leq \sum_{(u,v)\in\partial\Lambda_L} \sum_{(u',v')\in\partial\Lambda_{L+1}} \mathbb{E}\left|G_{\omega,L}(0,u;z)\right|^s \left|G_\omega(u',v;z)\right|^s \left|G_{\omega,L+1}(v',y;z)\right|^s$$

Compute expectation first in $\omega_{u'}$ and ω_v. Use independence and a priori bound. One gets

$$\mathbb{E}\left|G_\omega(0,y;z)\right|^s \leq C_s \sum_{(u,u')\in\partial\Lambda_L} \sum_{(v,v')\in\partial\Lambda_{L+1}} \mathbb{E}\left(\left|G_{\omega,L}(0,u;z)\right|^s\right)\mathbb{E}\left(\left|G_{\omega,L+1}(v',y;z)\right|^s\right)$$

$$\leq C_s' L^{(d-1)} L^d e^{-cL^{\frac{d}{d+2}}} \sum_{v'} \mathbb{E}\left|G_{\omega,L+1}(v',y;z)\right|^s \text{ ...iterate}$$

The estimate

$$\mathbb{E}\left(|G_{\omega,L}(0,u;z)|^s\right) \leq CL^d e^{-cL^{\frac{d}{d+2}}}$$

is a consequence of what is know as the "initial length estimate".
Write $E_0 = \inf \sigma(H_\omega)$ (in our case $E_0 = 0$).
We say

$$\Lambda\text{"is good"} \quad \Leftrightarrow \quad \inf \sigma(H_{\omega,\Lambda}) \geq E_0 + \frac{1}{L^\beta}.$$

$$\begin{aligned}
&\mathbb{E}\left(|G_{\omega,L}(0,u;z)|^s\right) \\
&= \mathbb{E}\left(|G_{\omega,L}(0,u;z)|^s \chi_{\{\Lambda\text{"is good"}\}}\right) + \mathbb{E}\left(|G_{\omega,L}(0,u;z)|^s \chi_{\{\Lambda\text{"is bad"}\}}\right)
\end{aligned}$$

Lemma (Initial length estimate)

For every $\beta \in (0,1)$ there are $\eta > 0$ and $C < \infty$ such that

$$\mathbb{P}\left(\inf \sigma(H_{\omega,\Lambda}) \leq E_0 + \frac{1}{L^\beta}\right) \leq CL^d e^{-\eta L^{\beta d/2}}$$

for all $L \in \mathbb{N}$.

Estimating the "initial lenght estimate" via IDS

Eigenvalue counting function : Let $\{\Lambda_L\}_{L\in\mathbb{N}}$ be a sequence of concentric cubes in $\mathbb{Z}^d$. Consider the restriction $H_\omega \restriction_{\Lambda_L} := \chi_{\Lambda_L} H_\omega \chi_{\Lambda_L}$. We define, for $E \in \mathbb{R}$,

$$N_L^\omega(E) := \frac{1}{\mathrm{vol}(\Lambda_L)} \sharp\{\text{e.v. of } H_\omega \restriction_{\Lambda_L} \leq E\}.$$

The Integrated Density of States (IDS) is defined as

$$N(E) := \lim_{L\to\infty} N_L^\omega(E).$$

• For the Anderson model H_ω on $\ell^2(\mathbb{Z}^d)$, as a consequence of ergodicity, we have

- ∗ Existence : the limit exists for $\mathbb{P}$-a.e. $\omega \in \Omega$, and is deterministic.
- ∗ Almost-sure spectrum : for $\mathbb{P}$-a.e. $\omega \in \Omega$,

$$\overline{\{E : E \text{ is a growth point of } N\}} = \sigma(H_\omega)$$

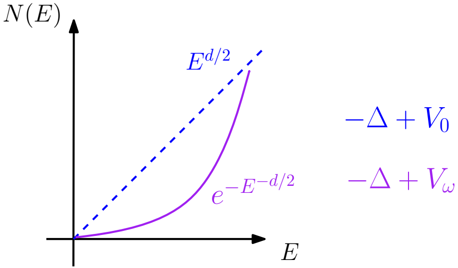
Lifshitz tails

Let $E_0 = \inf \sigma(-\Delta + V_0)$, with V_0 periodic. The Integrated Density of States (IDS) for $H = -\Delta + V_0$ behaves as

$$N(E) \sim (E - E_0)^{d/2}, \quad E \searrow E_0.$$

On the other hand, the IDS for the Anderson model $H_\omega = -\Delta + V_\omega$, behaves near $E_0 = \inf \Sigma$ as

$$N(E) \sim e^{-(E-E_0)^{-d/2}} \quad E \searrow E_0 \quad \text{Lifshitz tails}$$



For the Anderson model H_ω on $\ell^2(\mathbb{Z}^d)$,

- $*$ The IDS decays exponentially near the bottom of the spectrum $\Rightarrow$ localization.

How ?
Lifshitz tails $\Rightarrow$ Initial length estimate, i.e., existence of a spectral gap at the bottom of the spectrum of $H_{\omega,\Lambda}$, with good probability.

$$\mathbb{P}(\sigma(H_{\omega,L}) \cap [0,E]) \leq \mathbb{E}\left(\operatorname{tr} \chi_{[0,E]}(H_{\omega,L})\right) \text{`}\leq\text{'}\ CL^d N(E)$$
$$\leq cL^d e^{c' E^{-d/2}}$$

Take $E = \frac{1}{L^\beta}$ and obtain initial length estimate.
Now you can go on with the proof of localization using the Fractional Moment Method.
It works for energies near E_0 !

Beyond localization

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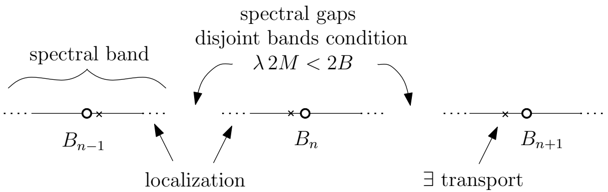
- There are a few results on delocalization, for very particular models. Ex. the Anderson model on the tree, and the Landau Hamiltonian in two dimensions.



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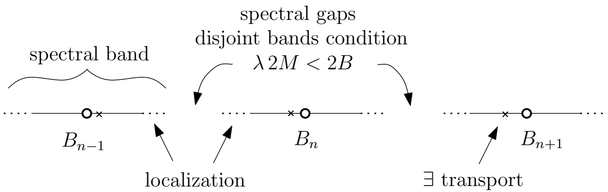

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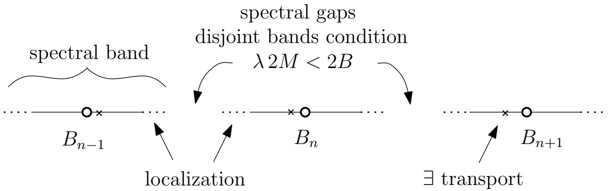

- The probability distribution of eigenvalues in the region of localization is typically Poisson.
- Many interacting particles. What is the right definition of localization ?

Thank you !

References

- G. Stolz, *An introduction to the mathematics of Anderson localization*, Contemporary Mathematics 551, 2010.
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