# UNIFORM CONVERGENCE OF DOUBLE VILENKIN-FOURIER SERIES

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# Definitions and notations

Let $\mathbb{N}_+$ denote the set of the positive integers, $\mathbb{N} := \mathbb{N}_+ \cup \{0\}$. Let $m := (m_{0,} m_{1....})$ denote a sequence of the positive integers not less than 2. Denote by $Z_{m_k} := Z \diagup m_k Z = \{[0], [1], ..., [m_k - 1]\}$ the addition group of integers modulo $m_k$.
Define the group $G_m$ as the complete direct product of the groups $Z_{m_i}$ with the product of the discrete topologies of $Z_{m_j}$ 's.
The direct product $\mu$ of the measures

$$\mu_k(\{j\}) := 1/m_k \qquad (j \in Z_{m_k})$$

is the Haar measure on $G_m$ with $\mu(G_m) = 1$.

# Definitions and notations

The elements of $G_m$ are represented by sequences

$$x := (x_0, x_1, ..., x_j, ...) \ \ (x_k \in Z_{m_k}) .$$

If the sequence $m$ is bounded then $G_m$ is called a bounded Vilenkin group, else it is called an unbounded one.
If we define the so-called generalized number system based on $m$ in the following way :

$$M_0 := 1, M_{k+1} := m_k M_k \ \ (k \in \mathbb{N}),$$

then every $n \in \mathbb{N}$ can be uniquely expressed as

$$n = \sum_{j=0}^{\infty} n_j M_j$$

# Definitions and notations

Next, we introduce on $G_m$ an ortonormal systems which are called the Vilenkin systems.
At first define the complex valued function $r_k(x) : G_m \to C$, The generalized Rademacher functions as

$$r_k(x) := \exp(2\pi i x_k / m_k) \quad \left(i^2 = -1, x \in G_m, k \in \mathbb{N}\right).$$

Now define the Vilenkin systems $\psi := (\psi_n : n \in \mathbb{N})$ on $G_m$ as:

$$\psi_n(x) := \prod_{k=0}^{\infty} r_k^{n_k}(x) \quad (n \in \mathbb{N}).$$

# Definitions and notations

The group

$$G_m^2 := G_m \times G_m$$

is called a two-dimensional Vilenkin group.

Two-dimensional systems: The Kronecker product $(\psi_{n,m} : n, m \in \mathbb{N})$ of two Vilenkin systems, where

$$\psi_{n,m}\left(x^1, x^2\right) = \psi_n\left(x^1\right)\psi_m\left(x^2\right).$$

Two-dimensional Vilenkin-Fourier coefficient:

$$\widehat{f}(n,m) := \int_{G_m^2} f\psi_{n,m} \quad (n, m \in \mathbb{N})$$

Rectangular partial sum of the Vilenkin-Fourier series

$$S_{n,m}(f; x^1, x^2) := \sum_{k=0}^{n-1}\sum_{i=0}^{m-1} \widehat{f}(k,i)\psi_{k,i}(x^1, x^2).$$

# Historical notes

Jordan C. Sur la series de Fourier. C.R. Acad. Sci. Paris. 92(1881), 228-230.

## Definition

We say that the function $f$ has Bounded variation and write $f \in BV$, if

$$V(f) < \infty.$$

## Theorem

*Let $f \in L_1$ and $f \in BV$. Then*

$$S_n f(x) \to (f(x+0) + f(x-0))/2, \ \text{when } n \to \infty.$$

# Historical notes

Hardy G. H. On double Fourier series and especially which represent the double zeta function with real and incommensurable parameters. Quart. J. Math. Oxford Ser. 37(1906), 53-79.

## Definition

We say that the function $f$ has Bounded variation in the sense of Hardy and write $f \in BV$, if

$$V(f) := V_1(f) + V_2(f) + V_{1,2}(f) < \infty.$$

## Theorem

*Let $f \in L_1$ and $f \in BV$. Then*

$$S_{n_1,n_2} f(x,y) \to \frac{1}{4} \sum f(x \pm 0, y \pm 0), \ \text{when } n_1, n_2 \to \infty.$$

# Historical notes

Goginava U. On the uniform convergence of multiple trigonometric Fourier series. East J. Approx. 3, 5(1999), 253-266.

## Definition

We say that the function $f$ has Bounded Partial variation and write $f \in PBV$, if

$$V(f) := V_1(f) + V_2(f) < \infty.$$

## Theorem

*Let $f \in L_1$ and $f \in PBV$. If limits $f(x \pm 0, y \pm 0)$ exist, then*

$$S_{n_1,n_2} f(x,y) \to \frac{1}{4} \sum f(x \pm 0, y \pm 0), \ \text{when } n_1, n_2 \to \infty.$$

# New result

Baramidze L., Uniform convergence of double Vilenkin-Fourier series, (in press).

## Theorem

*Let* $f \in C\left(G^2\right)$ *and the following conditions hold*

$$\lim_{k\to\infty} \sum_{\alpha=1}^{M_k-1} \frac{1}{\alpha} \left| \Delta_k^{(1)} f\left(x - z_\alpha^{(k)}, y\right) \right| = 0, \quad (1)$$

$$\lim_{l\to\infty} \sum_{\beta=1}^{M_l-1} \frac{1}{\beta} \left| \Delta_l^{(2)} f\left(x, y - z_\beta^{(l)}\right) \right| = 0, \quad (2)$$

$$\lim_{l,k\to\infty} \sum_{\alpha=1}^{M_k-1} \sum_{\beta=1}^{M_l-1} \frac{1}{\alpha} \frac{1}{\beta} \left| \Delta_{k,l}^{(1,2)} f\left(x - z_\alpha^{(k)}, y - z_\beta^{(l)}\right) \right| = 0 \quad (3)$$

*uniformly with respect to* $(x, y) \in G^2$. *Then the double Vilenkin-Fourier series of function* $f$ *converges uniformly on* $G^2$.

# New result

**Theorem**

*Let $f$ be a continuous function on $G^2$ and $f \in PBO\left(G^2\right)$. Then the Fourier series of $f$ converges uniformly on $G^2$.*

**Corollary**

*Let $f$ be a continuous function on $G^2$ and $f \in BO\left(G^2\right)$. Then the Fourier series of $f$ converges uniformly on $G^2$.*