K-theory for group C^* algebras

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Let J be an abelian semi-group.

$\widehat{J}$ denotes the abelian group :

$$\widehat{J} \quad = \quad J \bigoplus J / \sim$$

$$(\xi, \eta) \sim (\xi', \eta') \quad \Longleftrightarrow \quad \exists \quad \theta \in J \quad \text{with}$$

$$\xi + \eta' + \theta = \xi' + \eta + \theta$$

<u>Example</u> $\mathbb{N} = \{1, 2, 3, \ldots\}$

$$\widehat{\mathbb{N}} = \mathbb{Z}$$

Let Λ be a ring with unit 1_Λ.

$M_n(\Lambda)$ denotes the ring of all $n \times n$ matrices

$[a_{ij}]$ with each $a_{ij} \in \Lambda$. $\quad n = 1, 2, 3, \ldots$

$M_n(\Lambda)$ is again a ring with unit.

$GL(n,\Lambda) = \{$ invertible elements of $M_n(\Lambda)\}$

$P_n(\Lambda) = \{\alpha \in M_n(\Lambda) | \alpha^2 = \alpha\} \quad n = 1, 2, 3 \ldots$

Definition

$\alpha, \beta \in P_n(\Lambda)$ are similar if $\exists \gamma \in GL(n,\Lambda)$ with $\gamma\alpha\gamma^{-1} = \beta$.

Set $P(\Lambda) = P_1(\Lambda) \cup P_2(\Lambda) \cup P_3(\Lambda) \cup \ldots$

Impose an equivalence relation stable similarity on $P(\Lambda)$.

Definition

$\alpha \in P_n(\Lambda)$ and $\beta \in P_m(\Lambda)$ are stably similar iff there exist non-negative integers r, s with $n + r = m + s$ and with

$$\begin{array}{cc} \overbrace{\quad}^{n} & \overbrace{\quad}^{r} \end{array} \qquad\qquad \begin{array}{cc} \overbrace{\quad}^{m} & \overbrace{\quad}^{s} \end{array}$$

$$\begin{array}{|c|c|} \hline \alpha & 0 \\ \hline 0 & 0 \\ \hline \end{array} \text{ and } \begin{array}{|c|c|} \hline \beta & 0 \\ \hline 0 & 0 \\ \hline \end{array} \text{ similar}$$

Set $J(\Lambda) = P(\Lambda)/$(stable similarity).

$J(\Lambda) = P(\Lambda)/$(stable similarity)

$J(\Lambda)$ is an abelian semi-group.

$\alpha + \beta =$

α	0
0	β

Definition

$K_0\Lambda = \widehat{J(\Lambda)}$

This is the basic definition of K-theory.

Λ, Ω rings with unit

$\varphi\colon \Lambda \to \Omega$ ring homomorphism with $\varphi(1_\Lambda) = 1_\Omega$

$\varphi_*\colon K_0\Lambda \to K_0\Omega$

$\varphi_*[a_{ij}] = [\varphi(a_{ij})]$

$\varphi\colon K_0\Lambda \to K_0\Omega$ is a homomorphism of abelian groups

Example

If Λ is a field, then $[a_{ij}], [b_{kl}]$ in $P(\Lambda)$ are stably similar iff

$$\mathrm{rank}[a_{ij}] = \mathrm{rank}[b_{kl}],$$

where the rank of an $n \times n$ matrix is the dimension (as a vector space over Λ) of the sub vector space of $\Lambda^n = \Lambda \oplus \cdots \oplus \Lambda$ spanned by the rows of the matrix.

Hence if Λ is a field, $J(\Lambda) = \{0, 1, 2, 3, \ldots\}$ and $K_0\Lambda = \mathbb{Z}$.

X compact Hausdorff topological space

$C(X) = \{\alpha\colon X \to \mathbb{C} | \alpha \text{ is continuous}\}$

$C(X)$ is a ring with unit.

$(\alpha + \beta)x = \alpha(x) + \beta(x)$

$(\alpha\beta)x = \alpha(x)\beta(x) \qquad x \in X, \quad \alpha, \beta \in C(X)$

The unit is the constant function 1.

Definition (M. Atiyah - F. Hirzebruch)

Let X be a compact Hausdorff topological space.

$$K^0(X) = K_0 C(X)$$

Example

$$S^2 = \{(t_1, t_2, t_3) \in \mathbb{R}^3 \mid t_1^2 + t_2^2 + t_3^2 = 1\}$$

$$x_j \in C(S^2) \qquad x_j(t_1, t_2, t_3) = t_j \qquad j = 1, 2, 3$$

$$K_0 C(S^2) = \mathbb{Z} \oplus \mathbb{Z}$$

$$[1] \qquad \begin{bmatrix} \frac{1 + x_3}{2} & \frac{x_1 + ix_2}{2} \\ \frac{x_1 - ix_2}{2} & \frac{1 - x_3}{2} \end{bmatrix}$$

$i = \sqrt{-1}$


1, ε, ε², ε³, ε⁴
5 Eckp. des reg 5-Ecks
A5
ad - bc = 1,
a, b, c, d ∈ F5

C^* algebras

Definition

A Banach algebra is an algebra A over $\mathbb{C}$ with a given norm $\| \ \|$

$$\| \ \| : A \to \{t \in \mathbb{R} \mid t \geqq 0\}$$

such that A is a complete normed algebra:

$\|\lambda a\| = |\lambda| \|a\| \quad \lambda \in \mathbb{C}, \quad a \in A$

$\|a + b\| \leq \|a\| + \|b\| \quad a, b \in A$

$\|ab\| \leq \|a\| \|b\| \qquad a, b \in A$

$\|a\| = 0 \iff a = 0$

Every Cauchy sequence is convergent in A (with respect to the metric $\|a - b\|$).

C^* algebras

$A \quad C^*$ algebra $\qquad * \colon A \to A$

$A = (A, \| \, \|, *) \qquad a \mapsto a^*$

$(A, \| \, \|)$ is a Banach algebra

$(a^*)^* = a$

$(a+b)^* = a^* + b^*$

$(ab)^* = b^* a^*$

$(\lambda a)^* = \overline{\lambda} a^* \quad a, b \in A, \quad \lambda \in \mathbb{C}$

$\|aa^*\| = \|a\|^2 = \|a^*\|^2$

A $*$-homomorphism is an algebra homomorphism $\varphi \colon A \to B$ such that $\varphi(a^*) = (\varphi(a))^* \ \forall a \in A$.

Lemma

If $\varphi \colon A \to B$ is a $*$-homomorphism, then $\|\varphi(a)\| \leqq \|a\| \ \forall a \in A$.

EXAMPLES OF C^* ALGEBRAS

Example

X topological space, Hausdorff, locally compact

$$\begin{aligned}
X^+ &= \text{ one-point compactification of } X \\
&= X \cup \{p_\infty\} \\
C_0(X) &= \big\{\alpha\colon X^+ \to \mathbb{C} \mid \alpha \text{ continuous}, \alpha(p_\infty) = 0\big\} \\
\|\alpha\| &= \sup_{p \in X} |\alpha(p)| \\
\alpha^*(p) &= \overline{\alpha(p)} \\
(\alpha + \beta)(p) &= \alpha(p) + \beta(p) \quad p \in X \\
(\alpha\beta)(p) &= \alpha(p)\beta(p) \\
(\lambda\alpha(p) &= \lambda\alpha(p) \quad \lambda \in \mathbb{C}
\end{aligned}$$

If X is compact Hausdorff, then

$$C_0(X) = C(X) = \{\alpha\colon X \to \mathbb{C} \mid \alpha \text{ continuous}\}$$

Example

H separable Hilbert space
separable $=$ H admits a countable (or finite) orthonormal basis.

$$\mathcal{L}(H) = \{\text{bounded operators } T\colon H \to H\}$$

$$\|T\| = \sup_{\substack{u\in H\\ \|u\|=1}} \|Tu\| \qquad \underset{\|u\|=\langle u,u\rangle^{1/2}}{\text{operator norm}}$$

$$T^* = \text{ adjoint of } T \qquad \underset{u,v\in H}{\langle Tu, v\rangle = \langle u, T^* v\rangle}$$

$$(T+S)u = Tu + Su$$

$$(TS)u = T(Su)$$

$$(\lambda T)u = \lambda(Tu) \qquad \lambda \in \mathbb{C}$$

G topological group

locally compact

Hausdorff

second countable

(second countable = The topology of G has a countable base.)

Examples

Lie groups ($\pi_0(G)$ finite)	$\mathsf{SL}(n, \mathbb{R})$
p-adic groups	$\mathsf{SL}(n, \mathbb{Q}_p)$
adelic groups	$\mathsf{SL}(n, \mathbb{A})$
discrete groups	$\mathsf{SL}(n, \mathbb{Z})$

G topological group

locally compact

Hausdorff

second countable

Example

C_r^*G the reduced C^* algebra of G

Fix a left-invariant Haar measure dg for G

"left-invariant" $=$ whenever $f\colon G \to \mathbb{C}$ is continuous and compactly supported

$$\int_G f(\gamma g)dg = \int_G f(g)dg \qquad \forall \gamma \in G$$

L^2G Hilbert space

$L^2G = \{u\colon G \to \mathbb{C} \mid \int_G |u(g)|^2 dg < \infty\}$

$\langle u, v\rangle = \int_G \overline{u(g)} v(g) dg \qquad u, v \in L^2G$

$\mathcal{L}(L^2G) = C^*$ algebra of all bounded operators $T\colon L^2G \to L^2G$

$C_cG = \{f\colon G \to \mathbb{C} \mid f$ is continuous and f has compact support$\}$

C_cG is an algebra

$(\lambda f)g = \lambda(fg) \qquad \lambda \in \mathbb{C} \quad g \in G$

$(f+h)g = fg + hg$

Multiplication in C_cG is convolution

$$(f * h)g_0 = \int_G f(g)h(g^{-1}g_0)dg \qquad g_0 \in G$$

$0 \to C_c G \to \mathcal{L}(L^2 G)$

Injection of algebras

$f \mapsto T_f$

$T_f(u) = f * u \qquad u \in L^2 G$

$(f * u)g_0 = \int_G f(g)u(g^{-1}g_0)dg \qquad g_0 \in G$

$C_r^* G \subset \mathcal{L}(L^2 G)$

$C_r^* G = \overline{C_c G} =$ closure of $C_c G$ in the operator norm

$C_r^* G$ is a sub C^* algebra of $\mathcal{L}(L^2 G)$

A C^* algebra (or a Banach algebra) with unit 1_A.

Define abelian groups $K_1A, K_2A, K_3A, ...$ as follows :

$\mathrm{GL}(n, A)$ is a topological group.

The norm $\| \ \|$ of A topologizes $\mathrm{GL}(n, A)$.

$\mathrm{GL}(n, A)$ embeds into $\mathrm{GL}(n+1, A)$.

$$\mathrm{GL}(n, A) \hookrightarrow \mathrm{GL}(n+1, A)$$

$$\begin{bmatrix} a_{11} & ... & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & ... & a_{nn} \end{bmatrix} \mapsto \begin{bmatrix} a_{11} & ... & a_{1n} & 0 \\ \vdots & & \vdots & \vdots \\ a_{n1} & ... & a_{nn} & 0 \\ 0 & ... & 0 & 1_A \end{bmatrix}$$

$\mathrm{GL}\, A = \lim\limits_{n \to \infty} \mathrm{GL}(n, A) = \bigcup_{n=1}^{\infty} \mathrm{GL}(n, A)$

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Give $\mathrm{GL}\, A$ the direct limit topology.

This is the topology in which a set $U \subset \mathrm{GL}\, A$ is open if and only if $U \cap \mathrm{GL}(n, A)$ is open in $\mathrm{GL}(n, A)$ for all $n = 1, 2, 3, \ldots$

A C^* algebra (or a Banach algebra) with unit 1_A

$K_1 A, K_2 A, K_3 A, ...$

Definition

$K_j A := \pi_{j-1}(\mathrm{GL}\, A)$ $\qquad$ $j = 1, 2, 3, \ldots$

$\Omega^2\, \mathrm{GL}\, A \sim \mathrm{GL}\, A$ $\qquad$ Bott Periodicity

$K_j A \cong K_{j+2} A$ $\qquad$ $j = 0, 1, 2, \ldots$

$K_0 A$ $\qquad$ $K_1 A$

A C^* algebra (or a Banach algebra) with unit 1_A

$K_0 A = K_0^{alg} A = \widehat{J(A)}$

$A = (A, \| \|, *)$

For $K_0 A$ forget $\| \|$ and $*$. View A as a ring with unit.

Define $K_0 A$ as above using idempotent matrices.

For $K_1 A$ cannot forget $\| \|$ and $*$.

$K_0 A \qquad\qquad K_1 A$

A C^* algebra (or a Banach algebra) with unit 1_A

The Bott periodicity isomorphism

$K_0A = \widehat{J(A)} \longrightarrow K_2A = \pi_1 GLA$

assigns to $\alpha \in P_n(A)$ the loop of $n \times n$ invertible matrices

$t \mapsto I + (e^{2\pi i t} - 1)\alpha \qquad t \in [0, 1]$

$I =$ the $n \times n$ identity matrix

$A \quad C^*$ algebra (or a Banach algebra)

If A is not unital, adjoin a unit.

$0 \longrightarrow A \longrightarrow \tilde{A} \longrightarrow \mathbb{C} \longrightarrow 0$

Define: $K_j A = K_j \tilde{A} \qquad j = 1, 3, 5, \ldots$

$K_j A =$ Kernel$(K_j \tilde{A} \longrightarrow K_j \mathbb{C}) \qquad j = 0, 2, 4, \ldots$

$K_j A \cong K_{j+2} A \qquad j = 0, 1, 2, \ldots$

$K_0 A \qquad K_1 A$

FUNCTORIALITY OF K-THEORY

A, B $\qquad$ C^* algebras

$\varphi : A \longrightarrow B$ $\qquad$ $*$- homomorphism

$\varphi_* : K_j A \longrightarrow K_j B$ $\qquad$ $j = 0, 1$

G topological group

locally compact

Hausdorff

second countable

(second countable = topology of G has a countable base)

C_r^*G the reduced C^* algebra of G

Problem

$K_j C_r^* G = ?\quad j = 0, 1$

Conjecture (P. Baum - A. Connes)

$$\mu\colon K_j^G(\underline{E}G) \to K_j C_{exact}^* G$$

is an isomorphism. $j = 0, 1$

REMARK.
If G is exact, then

$$C^*_{exact}G = C^*_r G$$

The only known examples of non-exact groups are the Gromov groups
i.e. certain countable discrete groups which contain an expander in the Cayley graph.

All other locally compact groups (Lie groups, discrete groups, p-adic groups, adelic groups etc. etc.) are known to be exact.

So for all the groups that occur in “real life” can use $C^*_r G$ in the statement of BC.

G locally compact Hausdorff second countable topological group

Conjecture (P. Baum - A. Connes)

If G is an exact group (i.e. if G is not one of the Gromov groups), then

$$\mu\colon K_j^G(\underline{E}G) \to K_j C_r^* G$$

is an isomorphism. $j = 0, 1$

Γ discrete (countable) group

M C^∞-manifold, $\partial M = \emptyset$

$\Gamma \times M \to M$ smooth, proper, co-compact action of Γ on M.

"**smooth**" $=$ each $\gamma \in \Gamma$ acts on M by a diffeomorphism.

"**proper**" $=$ if Δ is any compact subset of M, then
$\{\gamma \in \Gamma : \Delta \cap \gamma\Delta \neq \emptyset\}$ is finite.

"**co-compact**" $=$ the quotient space M/Γ is compact.

Γ discrete (countable) group

Remarks

For a smooth proper co-compact action of Γ on M :

1. If $p \in M$, then $\{\gamma \in \Gamma : \gamma p = p\}$ is a finite subgroup of Γ.
2. M/Γ is a compact orbifold.
3. M is compact $\iff \Gamma$ is finite.

Γ discrete (countable) group

For the left side of BC,
shall now define abelian groups $K_j^{\text{top}}(\Gamma), \quad j = 0, 1$

Definition of $K_j^{\mathrm{top}}(\Gamma)$ $\qquad j = 0, 1$

Consider pairs (M, E) such that

1. M is a C^∞-manifold, $\partial M = \emptyset$, with a given smooth, proper co-compact action of Γ.

$$\Gamma \times M \to M$$

2. M has a given Γ-equivariant Spin^c-structure.
3. E is a Γ-equivariant $\mathbb{C}$ vector bundle on M.

$$K_0^{\mathrm{top}}(\Gamma) \oplus K_1^{\mathrm{top}}(\Gamma) = \{(M, E)\}/\sim$$

Addition will be disjoint union

$$(M, E) + (M, E') = (M \cup M', E \cup E')$$

Each fiber of E is a finite dimensional vector space over $\mathbb{C}$

$$\dim_{\mathbb{C}}(E_p) < \infty \quad p \in M$$

The equivalence relation
Isomorphism (M, E) is isomorphic to (M', E') iff $\exists$ a Γ-equivariant diffeomorphism

$$\psi \; M \to M'$$

preserving the Γ-equivariant Spin^c-structures on M, M' and with

$$\psi^*(E') \cong E$$

The equivalence relation $\sim$ will be generated by three elementary steps

- Bordism
- Direct sum - disjoint union
- Vector bundle modification

Bordism (M_0, E_0) is **bordant** to (M_1, E_1) iff $\exists$ (W, E) such that:

1. W is a C^∞ manifold with boundary, with a given smooth proper co-compact action of Γ

$$\Gamma \times W \to W$$

2. W has a given equivariant $\mathrm{Spin}^{\mathrm{c}}$-structure
3. E is a Γ-equivariant vector bundle on W
4. $(\partial W, E|_{\partial W}) \cong (M_0, E_0) \cup (-M_1, E_1)$

Direct sum - disjoint union
Let E, E' be two Γ-equivariant vector bundles on M

$$(M, E) \cup (M, E') \sim (M, E \oplus E')$$

Vector bundle modification

(M, E)

Let F be Γ-equivariant Spin^c vector bundle on M

Assume that

$$\dim_{\mathbb{R}}(F_p) \equiv 0 \mod 2 \quad p \in M$$

for every fiber F_p of F

$$\mathbf{1} = M \times \mathbb{R} \quad \gamma(p, t) = (\gamma p, t)$$

$$\gamma \in \Gamma \quad (p, t) \in \mathbf{1}$$

$$S(F \oplus \mathbf{1}) := \text{unit sphere bundle of } F \oplus \mathbf{1}$$

$$(M, E) \sim (S(F \oplus \mathbf{1}), \beta \otimes \pi^* E)$$

$$\begin{array}{c} S(F \oplus \mathbf{1}) \\ \Big\downarrow \pi \\ M \end{array}$$

This is a fibration with even-dimensional spheres as fibers
$F \oplus \mathbf{1}$ is a Γ-equivariant Spin^{c} vector bundle on M with odd dimensional fibers.

$$(M, E) \sim (S(F \oplus \mathbf{1}), \beta \otimes \pi^{*}E)$$

$$\{(M,E)\}/\sim = K_0^{\mathrm{top}}(\Gamma) \oplus K_1^{\mathrm{top}}(\Gamma)$$

$K_j^{\mathrm{top}}(\Gamma) =$ subgroup of $\{(M,E)\}/\sim$ consisting of all (M,E) such that every connected component of M has dimension $\equiv j \mod 2,\ j = 0, 1$

Notation: for $(M, E), D_E$ is the Dirac operator of M tensored with E

$F =$ spinor bundle of M

$D_E : C_c^\infty(M, F \otimes E) \to C_c^\infty(M, F \otimes E)$

$$K_j^{\mathrm{top}}(\Gamma) \to K_j(C_r^*\Gamma) \quad j = 0, 1$$

$$(M, E) \mapsto \mathrm{Index}(D_E)$$

Conjecture (BC). (P. Baum, A. Connes)

For any countable discrete exact group Γ

$$K_j^{\mathrm{top}}(\Gamma) \to K_j(C_r^*\Gamma) \quad j = 0, 1$$

is an isomorphism

Corollary

If BC conjecture is true for Γ, then

1. *Every element of $K_j(C_r^*\Gamma)$ is of the form $\mathrm{Index}(D_E)$ for some (M, E) (surjectivity)*
2. (M, E) *and* (M', E') *have*

$$\mathrm{Index}(D_E) = \mathrm{Index}(D'_{E'})$$

if and only if it is possible to pass from (M, E) to (M', E') by a finite sequence of the three elementary moves

- *Bordism*
- *Direct sum - disjoint union*
- *Vector bundle modification*

(injectivity)

Example. Let Γ be a finite group. Consider

$$K_0^{\text{top}}(\Gamma) \to K_0(C_r^*\Gamma)$$
$$(M, E) \mapsto \text{Index}(D_E)$$

when Γ is a finite group.

Since Γ is a finite group, the M in any (M, E) is compact. Therefore

$$D_E : C^\infty(\mathcal{S}^+ \otimes E) \to C^\infty(\mathcal{S}^- \otimes E)$$

has finite dimensional kernel and cokernel.
Then Index (D_E):= kernel(D_E) - cokernel(D_E) $\in$ R(Γ).
R(Γ):= the representation ring of Γ.

Γ a finite group. R(Γ):= the representation ring of Γ.

R(Γ) is a free abelian group with one generator for each equivalence class of irreducible representations (on $\mathbb{C}$ vector spaces) of Γ.

$$K_0^{\text{top}}(\Gamma) \cong \mathsf{R}(\Gamma)$$
$$K_1^{\text{top}}(\Gamma) = 0$$

This is proved using Γ-equivariant Bott periodicity (i.e. same as proof in lecture 2 that $K_0(\cdot) = \mathbb{Z}$).

BC (and BC with coefficients) are for topological groups G which are locally compact, Hausdorff, and second countable.

$\underline{E}G$ denotes the universal example for proper actions of G.

EXAMPLE. If Γ is a (countable) discrete group, then $\underline{E}\Gamma$ can be taken to be the convex hull of Γ within $l^2(\Gamma)$.

Example

Give Γ the measure in which each $\gamma \in \Gamma$ has mass one.
Consider the Hilbert space $l^2(\Gamma)$.
Γ acts on $l^2(\Gamma)$ via the (left) regular representation of Γ.
Γ embeds into $l^2(\Gamma)$ $\qquad \Gamma \hookrightarrow l^2(\Gamma)$
$\gamma \in \Gamma \qquad \gamma \mapsto [\gamma]$ where $[\gamma]$ is the Dirac function at γ.
Within $l^2(\Gamma)$ let Convex-Hull(Γ) be the smallest convex set which contains Γ. The points of Convex-Hull(Γ) are all the finite sums

$$t_0[\gamma_0] + t_1[\gamma_1] + \cdots + t_n[\gamma_n]$$

$$\text{with} \quad t_j \in [0,1] \quad j = 0, 1, \ldots, n \quad \text{and} \quad t_0 + t_1 + \cdots + t_n = 1$$

The action of Γ on $l^2(\Gamma)$ preserves Convex-Hull(Γ).
$\Gamma \times$Convex-Hull(Γ) $\longrightarrow$Convex-Hull(Γ)
$\underline{E}\Gamma$ can be taken to be Convex-Hull(Γ) with this action of Γ.

Let X be a paracompact Hausdorff topological space with a given continuous action of G on X.

$$G \times X \longrightarrow X$$

The action of G on X is proper if :

The quotient space X/G (with the quotient topology) is paracompact Hausdorff and

For each $x \in X, \exists$ a triple (U, H, φ) such that:

- U is an open set of X with $x \in U$ and with $gp \in U$ whenever $g \in G$ and $p \in U$.
- H is a compact subgroup of G.
- $\varphi\colon U \to G/H$ is a continuous G-equivariant map from U to G/H.

$\underline{E}G$ is a paracompact Hausdorff topological space with a given proper action of G :

$$G \times \underline{E}G \longrightarrow \underline{E}G$$

such that whenever X is a paracompact Hausdorff topological space with a given proper action of G on X

- $\exists$ a continuous G-equivariant map $f\colon X \to \underline{E}G$.
- Any two continuous G-equivariant maps $f_0\colon X \to \underline{E}G$, $f_1\colon X \to \underline{E}G$ are homotopic through continuous G-equivariant maps.

Examples of $\underline{E}G$

G compact, $\quad \underline{E}G = \cdot$

G a Lie group with $\pi_0(G)$ finite $\quad \underline{E}G = G/K$ where K is a maximal compact subgroup of G.

G a reductive p-adic group $\underline{E}G =$ the affine Bruhat-Tits building of G.

n, m two positive integers $G = (\mathbb{Z}/n\mathbb{Z}) * (\mathbb{Z}/m\mathbb{Z})$
$\underline{E}(G) =$ the tree on which G acts. See book Trees by J. P. Serre.

$K_j^G(\underline{E}G)$ denotes the Kasparov equivariant K-homology — with G-compact supports — of $\underline{E}G$.

Definition

A closed subset Δ of $\underline{E}G$ is G-compact if:
1. The action of G on $\underline{E}G$ preserves Δ.
and
2. The quotient space Δ / G (with the quotient space topology) is compact.

Definition

$$K_j^G(\underline{E}G) = \varinjlim_{\substack{\Delta \subset \underline{E}G \\ \Delta\ G\text{-compact}}} KK_G^j(C_0(\Delta), \mathbb{C}).$$

The direct limit is taken over all G-compact subsets Δ of $\underline{E}G$. $K_j^G(\underline{E}G)$ is the Kasparov equivariant K-homology of $\underline{E}G$ with G-compact supports.

BC conjecture for exact groups

Conjecture

For any G which is locally compact, Hausdorff, second countable, and exact

$$K_j^G(\underline{E}G) \to K_j(C_r^*G) \qquad j = 0, 1$$

is an isomorphism

BC conjecture in general i.e. including non-exact groups

Conjecture

For any G which is locally compact, Hausdorff, and second countable

$$K_j^G(\underline{E}G) \to K_j(C^*_{exact}G) \qquad j = 0, 1$$

is an isomorphism

Corollaries of BC

Novikov conjecture $=$ homotopy invariance of higher signatures

Stable Gromov Lawson Rosenberg conjecture (Hanke + Schick)

Idempotent conjecture

Kadison Kaplansky conjecture

Mackey analogy (Higson)

Exhaustion of the discrete series via Dirac induction
(Parthasarathy, Atiyah + Schmid, V. Lafforgue)

Homotopy invariance of ρ-invariants
(Keswani, Piazza + Schick)

G topological group
locally compact, Hausdorff, second countable

Examples

Lie groups ($\pi_0(G)$ finite)	$\mathsf{SL}(n,\mathbb{R})$ OK✓
p-adic groups	$\mathsf{SL}(n,\mathbb{Q}_p)$OK✓
adelic groups	$\mathsf{SL}(n,\mathbb{A})$OK✓
discrete groups	$\mathsf{SL}(n,\mathbb{Z})$

Let A be a $G - C^*$ algebra i.e. a C^* algebra with a given continuous action of G by automorphisms.

$$G \times A \longrightarrow A$$

BC with coefficients for exact groups

Conjecture

For any G which is locally compact, Hausdorff, and second countable and any $G - C^*$ algebra A

$$K_j^G(\underline{E}G, A) \to K_j(C_r^*(G, A)) \qquad j = 0, 1$$

is an isomorphism.

Let A be a $G - C^*$ algebra i.e. a C^* algebra with a given continuous action of G by automorphisms.

$$G \times A \longrightarrow A$$

BC with coefficients in general i.e. including non-exact groups

Conjecture

For any G which is locally compact, Hausdorff, and second countable and any $G - C^*$ algebra A

$$K_j^G(\underline{E}G, A) \to K_j(C^*_{exact}(G, A)) \qquad j = 0, 1$$

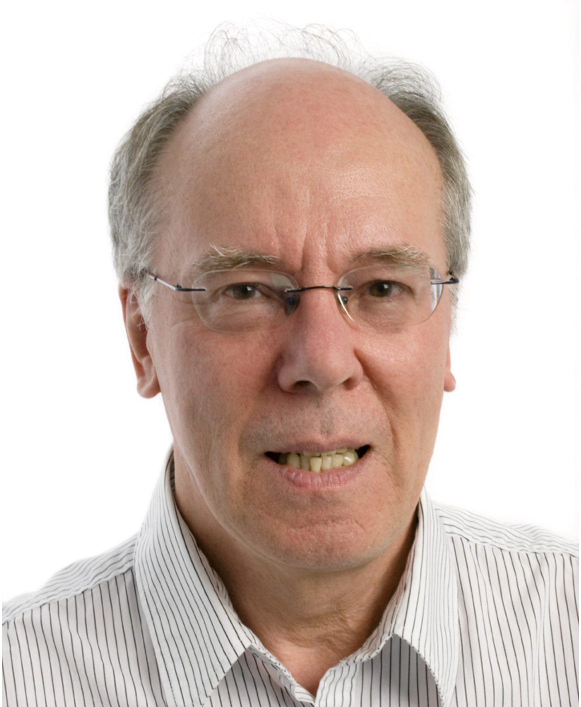
is an isomorphism.

Definition

$$K_j^G(\underline{E}G, A) = \varinjlim_{\substack{\Delta \subset \underline{E}G \\ \Delta\ G\text{-compact}}} KK_G^j(C_0(\Delta)\,,\, A).$$

The direct limit is taken over all G-compact subsets Δ of $\underline{E}G$.

$K_j^G(\underline{E}G, A)$ is the Kasparov equivariant K-homology of $\underline{E}G$ with G-compact supports and with coefficient algebra A.



Theorem (N. Higson + G. Kasparov)

Let Γ be a discrete (countable) group which is amenable or a-t-menable. Let A be any $\Gamma - C^*$ algebra. Then

$$\mu\colon K_j^{\Gamma}(\underline{E}\Gamma, A) \to K_j C_r^*(\Gamma, A)$$

is an isomorphism. $j = 0, 1$

Theorem (G. Yu + I. Mineyev, V. Lafforgue)

Let Γ be a discrete (countable) group which is hyperbolic (in Gromov's sense). Let A be any $\Gamma - C^*$ algebra. Then

$$\mu\colon K_j^{\Gamma}(\underline{E}\Gamma, A)) \to K_j C_r^*(\Gamma, A)$$

is an isomorphism. $j = 0, 1$

$\mathrm{SL}(3, \mathbb{Z})$??????